On an elliptic system with symmetric potential possessing two global minima

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Abstract

We consider the system

$$\Delta u - W_u(u) = 0, \text{ for } u : \mathbb{R}^2 \to \mathbb{R}^2, \ W : \mathbb{R}^2 \to \mathbb{R},$$

where $W_u(u) = \big(W_{u_1}(u), W_{u_2}(u)\big)$, in an equivariant class of functions. We prove that there exists u, a two-dimensional solution, which satisfies the conditions

$$u(x_1, x_2) \to a^{\pm}, \text{ as } x_1 \to \pm\infty,$$

where $a^+, a^- \in \mathbb{R}^2$ are the two global minima of the potential W. We also consider the problem on bounded rectangular domains with Neumann boundary conditions and also give higher-dimensional extensions. The problem above was first studied by Alama, Bronsard, and Gui in [1].

1 Introduction

We begin by describing the general context. The problem we examine is a simple but representative example. The objects of study are primarily certain entire solutions to

$$(1.1) \qquad \Delta u - W_u(u) = 0, \text{ for } u : \mathbb{R}^N \to \mathbb{R}^N, \ W : \mathbb{R}^N \to \mathbb{R}.$$

Here, W is a potential that has a finite number of (global) minima, also called 'phases', M$=$ $\{a_1, \ldots, a_n\}$, with $W(a_1) = \cdots = W(a_n) = 0$ and $W(u) > 0$ otherwise. We note that (1.1) includes solutions

$$u : \mathbb{R}^i \to \mathbb{R}^N, \text{ for } 1 \le i < N,$$

by trivial extension. The entire solutions we are seeking are bounded and, in addition, they approach the minima of W in certain directions at infinity. For example, if $i = 1$, then (1.1) becomes a Hamiltonian system of second-order ODEs and the solutions of interest are those satisfying the 'boundary' conditions

$$u(x_1) \to a^{\pm}, \text{ as } x_1 \to \pm\infty.$$

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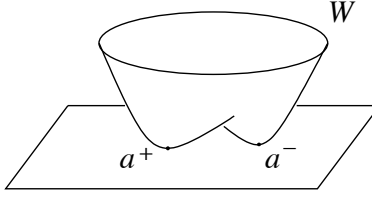
Figure 1: The potential W with two minima a^+, a^-.

Such solutions for $i = 1$ are known as *heteroclinics*, and the corresponding problem is known as the connection problem [19].

Another important hypothesis on W is symmetry. We assume that

$$W(gu) = W(u), \text{ for all } g \in G,$$

where G is a finite reflection subgroup of the orthogonal group $O(\mathbb{R}^N)$, and we seek G-*equivariant* solutions to (1.1), that is, solutions satisfying

$$u(gx) = gu(x), \text{ for all } g \in G.$$

The *fundamental region* F for a finite subgroup of $O(\mathbb{R}^N)$ is a convex set, actually it is the intersection of half-spaces [10]. It is defined in Section 3, and it plays a role in our considerations. For the main example in this paper, $N = n = 2$, G=$\mathcal{H}_2^2$, the dihedral group with four elements $\{I, T_1, T_2, S\}$ (the two reflections with respect to the axes u_1 and u_2, the rotation by π, and the identity), while F in this case is $\{(u_1, u_2)|u_1 \geq 0, u_2 \geq 0\}$ and $a^{\pm}=(\pm a, 0)$ for $a > 0$.

Problem (1.1) has variational structure. It is clearly the Euler–Lagrange equation corresponding to the functional

$$J(u) = \int_{R^N} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dx.$$

An important feature of the problem and also a source of difficulty is that for the solutions we are seeking, for $N > 1$, the functional J is not finite. For $N = 1$, J coincides with the action; in this case, it is finite and heteroclinic solutions or connections can be obtained as minimizers of the action in the class of functions approaching $a^{\pm}$ at $\pm\infty$ (see [2] for the scalar case and [5] for the vector one).

The problem at hand is of interest because it is nonconvex and also a system. Moreover, the class of solutions sought are not radial. It originates from geometric evolution and phase transitions. The relevant dynamic problem $u_t = \varepsilon^2 \Delta u - W_u(u)$, on $\mathbb{R}^N$, is known as the vector Allen-Cahn equation; this is a gradient flow of the functional

$$\int_{\mathbb{R}^N} \left\{ \frac{1}{2}|\varepsilon \nabla u|^2 + W(u) \right\} dx.$$

For small positive ε, one expects that typical initial data will evolve quickly, at an O(1) time-scale, towards the set M of wells, so that the domain is partitioned into *phase regions*, in each

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one of which u is approximately constant ($u \simeq a_i$). These regions are separated by thin zones, the *diffused interfaces*, that evolve approximately by the geometric law $V = \varepsilon^2 H$. Here, V is the normal velocity and H the mean curvature of the interface. The profile of the solution near the interfaces is a rescaled connection. At a *junction*, the meeting point of the interfaces, the angles attain certain values (the *Plateau conditions*) that remain fixed as long as the junction exists. For example, for $N = 2$ and a triple-well potential, typically, triple-junctions are formed. The structure of u near the interface and away from the triple-junction is essentially one-dimensional, depending only on the distance from the interface, with a profile close to a scaled version of a solution to (1.1) with $i = 1$, $N = 2$. This is plausible since at the interface the Laplacian and the free term balance each other, to principal order in ε, and thus (1.1) is satisfied approximately. The angle conditions are determined by the transition energies connecting the states on each side of the interface. On the other hand, at the junction the structure is two-dimensional and is close to a scaled solution of (1.1) with $i = 2$, $N = 2$. We refer to [8] for formal and rigorous evidence supporting these scenarios. The rigorous analysis of the PDE solution at the junction was done subsequently in [7] for a potential respecting the symmetries of the equilateral triangle.

The main example of the present paper is also motivated from the dynamics of interfaces. As was established in [3], there are multiple-well potentials W for which the connection problem between two phases admits two (or more) distinct solutions. To be specific, assume that there exist precisely two connections $e_\pm$ connecting two of the wells of the potential. As a result, the interface separating these two phases is made up of two types of pieces (see Figure 1 in [3]). Simulations show that a wave is generated on the interface itself, which propagates and converts the interface into the type with lesser action. The structure of the solution at the boundary between these pieces is two-dimensional, and well approximated by a rescaled solution to

$$
\text{(1.2)} \quad
\begin{cases}
\Delta u - W_u(u) = -c \dfrac{\partial u}{\partial x_2}, \text{ for } u : \mathbb{R}^2 \to \mathbb{R}^2, \\[2mm]
u(x_1, x_2) \to a^\pm, \text{ as } x_1 \to \pm\infty, \\[2mm]
u(x_1, x_2) \to e_\pm(x_1), \text{ as } x_2 \to \pm\infty.
\end{cases}
$$

Problem (1.2) is a travelling-wave problem with speed c. If e_+ and e_- have equal actions, then $c = 0$ and the speed of the wave, to principal order, is zero. In simulations we observe propagation also in this case but now slower and with a speed apparently determined by geometric effects. In the present paper we study (1.2) with $c = 0$. Problem (1.2) for $c \neq 0$ is, to our knowledge, still open [9].

Returning back to the discussion of triple-junctions on the plane, we note that there is an analog in three dimensions. W now has to be a quadruple-well potential ($n = 4$). The interfaces become surfaces and their meeting point is a quadruple-junction where the four phases coexist. At the junction, the solution is approximated by a rescaled solution to (1.1) with $N = 3$. This has been established at the level of formal asymptotics in [4]; the rigorous analysis has appeared very recently in [11].

The discussion above suggests that there are three important numbers in the study of (1.1) that can be singled out.

- The *number of phases*, which equals the number of wells, that is, the number of minima.

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- The *minimal dimension* in the u-space that allows coexistence of a given number of phases. For example, for coexistence of three phases, u has to be two-dimensional.

- The *genuine dimension* of the solution to (1.1), that is, the minimal i in (1.1) that describes the solution.

We now come to the statement of our main results. We begin with the hypotheses.

(h$_1$) The potential W is C^2, $W : \mathbb{R}^2 \to \mathbb{R}_+ \cup \{0\}$, and has exactly two nondegenerate global minima $a^\pm$, where $a^+ = (a, 0)$, $a^- = (-a, 0)$, for $a > 0$. Also, $\partial^2 W(u) \geq c^2 \mathrm{Id}$, for $|u - a^\pm| < r_0$.

(h$_2$) W has the symmetry of the dihedral group $\mathcal{H}_2^2$. Thus $W(gu) = W(u)$, $g \in \mathcal{H}_2^2$. The solution u is $\mathcal{H}_2^2$-equivariant, that is, $u(g(x)) = g(u(x))$, $g \in \mathcal{H}_2^2$. Finally, we assume that $W(u) \geq \max_{\partial C_0}[W]$, for u outside a certain bounded, $\mathcal{H}_2^2$-symmetric, convex set C_0.

(h$_3$) We set $D := \{(u_1, u_2) \mid u_1 \geq 0\}$. We assume that there exists a C^2 function $Q : D \setminus \{a^+\} \to \mathbb{R}_+ \cup \{0\}$, convex, with $Q(u) > 0$ for $u \in D \setminus \{a^+\}$, $Q(u) = |u - a^+|$ for $|u - a^+| < r_0$, satisfying the relation

$$W_u(u) \cdot Q_u(u) \geq 0, \text{ for } u \in D.$$

(h$_4$) The 'scalar' trajectory e_0 which always exists by symmetry[1] and as a curve lies on the u_1 axis and connects a^+, a^-, is assumed not to be a global minimum of the action

$$E(U) = \int_{\mathbb{R}} \left\{ \frac{1}{2} |U_x|^2 + W(U) \right\} \mathrm{d}x$$

among the trajectories connecting a^- and a^+. It follows by [5] that there exists at least one pair of connecting trajectories $e_\pm$ which globally minimize the action in the class of trajectories that connect $a^\pm$ and with action strictly less than that of the scalar trajectory, $E(e_\pm) < E(e_0)$. We denote the set of minimizing connections of the action by A.

Under the hypotheses above, we establish the following

Theorem 1.1. *There exists a solution u to*

$$\Delta u - W_u(u) = 0, \text{ for } u : \mathbb{R}^2 \to \mathbb{R}^2,$$

which is $\mathcal{H}_2^2$-equivariant and satisfies the estimate

$$|u(x) - a^+| < M\mathrm{e}^{-c|x_1|}, \text{ for } x_1 \geq 0,$$

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[1]The symmetry of W assumed in (h$_2$) implies that $W_{u_2}(u_1, 0) = 0$. Consequently, the solution of the scalar equation $e_{x_1 x_1} - W_{u_1}(e, 0) = 0$, $e(\pm\infty) = \pm a$, extends trivially to a solution of (1.1) by setting $e_0(x_1) = (e(x_1), 0)$. We normalize it by taking $e(0) = 0$.

where M is a constant depending only on W and c is as in (h$_1$), which in particular implies that

(1.3)
$$u(x) \not\equiv 0.$$

Also,

(1.4)
$$u(x) \not\equiv e_0(x_1), \quad (e_0(0) = 0)$$

where $e_0(x_1)$ is the scalar connection introduced in (h$_4$).

Moreover the solution is genuinely two-dimensional[2]: there exists a sequence $x_2^n \to \infty$, such that

(1.5)
$$u(x_1, x_2^n) \to \tilde{e}_+(x_1) \quad \text{and} \quad u(x_1, -x_2^n) \to \tilde{e}_-(x_1),$$

where $\tilde{e}_\pm$ are connecting orbits of a^+, a_-, symmetric to each other, with $\tilde{e}_\pm(0) = 0$, and distinct from the scalar e_0.

The theorem above is modeled after a similar result in Alama, Bronsard, and Gui [1]. In spite of the many similarities between the two results, there are also significant differences that stem mainly from the hypotheses and the methods of proof. In [1], the authors consider an expanding sequence of infinite horizontal strips and impose Dirichlet conditions that effectively tie the solution to a given pair of connections in the x_2-direction and to the two minima in the x_1-direction. We instead consider a unilateral constraint only in the x_1-direction that forces the solution to be close to the minima at $\pm\infty$ and otherwise minimize freely, thus solving a Neumann problem on each strip. We derive information in the x_2-direction *a posteriori*, and at the very end. As a result, for example, contrary to [1], we do not need to assume uniqueness of the pair of connections $e_\pm$ (see Example 2 below) and thus allow, in principle, the existence of multiple solutions.

Easy estimates allow the passage to a limiting u that satisfies the equation in the whole plane. The task then is to show that u is not trivial. Equivariance excludes the zero-dimensional solutions $u \equiv a^\pm$ and also excludes one-dimensional solutions like $e_\pm$. For excluding the possibility $u \equiv 0$, we utilize a uniform (with respect of the expanding domains) exponential estimate which is among the main tools in our work. Here we invoke (h$_3$), the Q-monotonicity of W. For excluding the scalar connection e_0, we utilize (h$_4$).

The other major point is a positivity result that underlies the whole procedure: we show that the minimizers of the constrained problems leave the fundamental region invariant.

In passing to the limit, we follow an idea from [1] that gives tight upper and lower bounds (see Steps 1, 2 and 3 in the proof of Theorem 7.1). We note that [1] follows closely the methodology of [7]. The limiting procedure along strips can be found in [15], [16].

Our next theorem concerns the bounded-domain problem on large rectangles.

Theorem 1.2. *Suppose that W satisfies hypotheses (h$_1$), (h$_2$). Then, we can determine $R_0 \geq 1$ and $\mu_0 \geq 1$, so that for $R > R_0$ and $\mu \geq \mu_0$, there exists an equivariant minimizer of*

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[2]See the discussion following (1.1); u cannot be described by a function with $i = 1$.

the functional $J_{R,\mu}(u) = \int_{\Omega_{R,\mu}} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} \mathrm{d}x$ that satisfies, weakly in $W^{1,2}(\Omega_{R,\mu})$, the problem

(1.6)
$$\begin{cases} \Delta u - W_u(u) = 0, & in \ \Omega_{R,\mu}, \\ \dfrac{\partial u}{\partial n} = 0, & on \ \partial \Omega_{R,\mu}. \end{cases}$$

Moreover, if in addition (h_3), (h_4) *are satisfied, then the solution tends, as* $R \to \infty$, *to a solution as described in* (1.5) *in Theorem 1.1. Here,* $\Omega_{R,\mu} = \big\{ (x_1, x_2) \mid |x_1| < \mu R, \ |x_2| < R \big\}$. *Thus, it is genuinely two-dimensional.*

The paper is organized as follows. In Section 1 we give two examples that fit in the framework of Theorems 1.1 and 1.2, and two higher-dimensional extensions of these theorems. The first extension is a three-dimensional result whose purpose is to bring out the hierarchy of the solutions that can exist in the same problem. Such hierarchies have been found already in [11]. The second extension is an N-dimensional result closer in spirit to Theorem 1.1, together with an application to a system that possesses an entire radial solution. In the rest of the paper we give the proofs. Specifically, in Section 3 we introduce and solve the constrained problem, in Section 4 we establish the positivity property, and in Section 5 we establish half of the bounded-domain theorem (existence) by showing that the constraint is not realized for large domains. In Section 6 we establish the exponential estimate by an iteration argument and finally, in Section 7 we take the limit and establish Theorem 1.1 from which the remaining of Theorem 1.2 (genuine two-dimensionality) follows. In Section 8 we give the proofs of the higher-dimensional extensions.

An important question is to what extent the approach followed here and consists of the constrained problem, the positivity property (establishes low complexity of the minimizer), and the exponential estimate, is capable of generalization to the general finite reflection group. The exponential estimate is technically easy in the present setup because of the simplicity of the geometry. In [6], we have been able to obtain the exponential estimate in a general setup under (h_3). The positivity property appears hard to extend to the general case. However, the symmetries of the group impose boundary conditions on the fundamental domain. By utilizing the parabolic flow, we have been able to show that there is a minimizer in the class of the positive maps [6] that does not realize the constraint. Thus, overall, the approach appears general.

2 Observations, examples, and higher-dimensional extensions

Example 1. Consider the potential

(2.1)
$$W_1(z) = \left| \frac{z^2 - 1}{z^2 + \varepsilon^2} \right|^2, \ \text{for } 0 < \varepsilon < \infty, \ z = u_1 + iu_2, \ u = (u_1, u_2).$$

W_1 has two global minima at $a^\pm = (\pm 1, 0)$ and obviously has the symmetry (h_2). It has been shown in [3] that there exist exactly three trajectories connecting -1 with 1, e_+^ε, e_-^ε, and e_0^ε, with e_+^ε, e_-^ε reflections of each other with respect to the u_1-axis and with e_0^ε lying
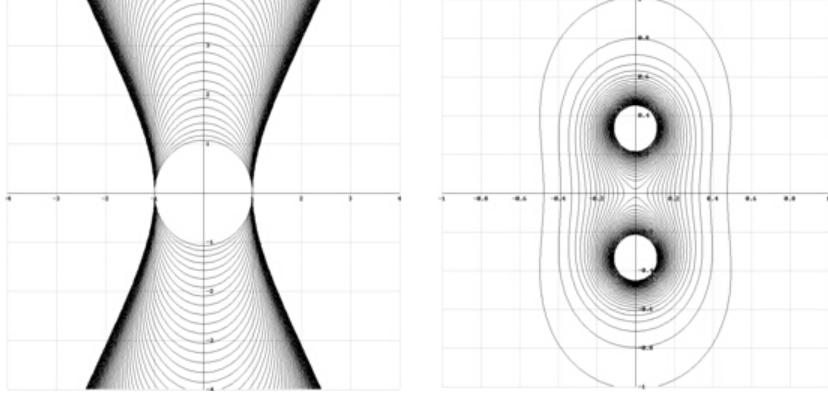
Figure 2: The figure on the left shows a computation of the trajectories $e_\pm^\varepsilon$ for the potential W_1, for $0 < \varepsilon < \infty$. We note that $e_\pm^\varepsilon$ tend to the unit circle as $\varepsilon \to 0$ while their envelope, as $\varepsilon \to \infty$, is given by $u_1^2 = u_2^2/3 + 1$. The disc-like boundary shown in the figure corresponds to $\varepsilon = \sqrt{3}/6 < \varepsilon^* = 0.4416\ldots$ The region bounded by $e_\pm^\varepsilon$ ceases to be convex for $\varepsilon = \sqrt{3}$ [14]. On the right are the level sets of $W_1(z)$ for $\varepsilon = \sqrt{3}/6 < \varepsilon^* = 0.4416\ldots$ The existence of a Q such that $Q_u \cdot W_u \geq 0$ in D is geometrically evident [14].

on the u_1-axis (see Figure 2a). Moreover, $E(e_\pm^\varepsilon) < E(e_0^\varepsilon)$ for $0 < \varepsilon < \varepsilon^* = 0.4416\ldots$ and $E(e_\pm^\varepsilon) > E(e_0^\varepsilon)$ for $\varepsilon > \varepsilon^*$. In more detail, the trajectories $e_\pm^\varepsilon$ are determined by the equation

$$u_2 + \frac{1+\varepsilon^2}{4\varepsilon} \ln\left(\frac{(u_2 - \varepsilon)^2 + u_1^2}{(u_2 + \varepsilon)^2 + u_1^2}\right) = 0$$

and

$$E(e_0^\varepsilon) = \frac{1}{\sqrt{2}}\left(\frac{1+\varepsilon^2}{\varepsilon}(\pi - \arctan\varepsilon) - \varepsilon\right), \quad E(e_\pm^\varepsilon) = \frac{1}{\sqrt{2}}\left(2 + \frac{2(1+\varepsilon^2)}{\varepsilon}\arctan\varepsilon\right).$$

Modifying W_1 near the poles $\pm\varepsilon i$ allows us to produce a C^∞ potential $\tilde{W}$ possessing the above trajectories. Clearly, the potential $\tilde{W}$ satisfies the hypotheses (h$_1$), (h$_2$), (h$_4$). For explaining the Q-monotonicity of W, condition (h$_3$), we consider for the moment the hypothesis

(h$_3^*$) $W_u(u) \cdot (u - a^+) \geq 0,$ for $u \in D.$

(h$_3^*$) corresponds to the choice $Q(u) = |u - a^+|$ and states the monotonicity of W along rays emanating from a^+. In the one-dimensional case, the hypotheses (h$_3$), (h$_3^*$) coincide.

   In higher dimensions however, (h$_3$) is significantly weaker than (h$_3^*$). For example, in the case when W is a center at the origin, (h$_3^*$) is never satisfied. In contrast, the existence of a convex Q which satisfies (h$_3$) appears very plausible. Theorem 7.1 produces an entire solution which appears to map the plane into the region bounded by the two symmetric connections. We believe that in this case the entire solution is unique, and moreover, a global diffeomorphism of the plane onto this region.

Example 2. Consider the potential

(2.2) $W_2(z) = \left|\dfrac{z^2 - 1}{z^2 + \varepsilon_1^2}\right|^2 \left|\dfrac{z^2 - 1}{z^2 + \varepsilon_2^2}\right|^2,$ for $0 \leq \varepsilon_1 \leq \varepsilon_2 < \infty,$ $z = u_1 + iu_2,$ $u = (u_1, u_2).$
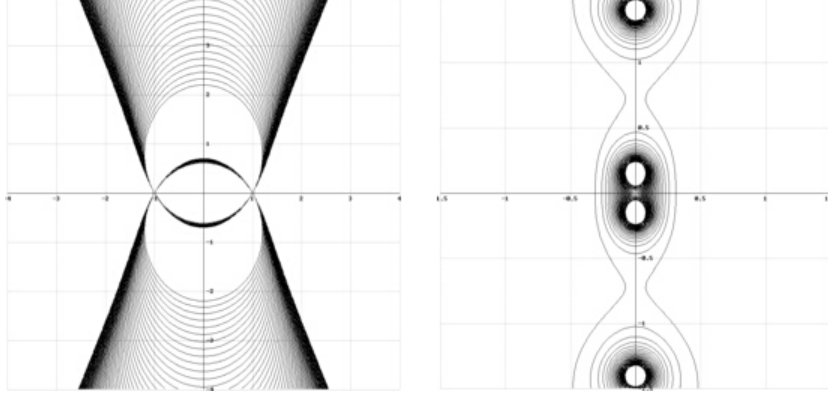
Figure 3: The figure on the left shows a computation of the trajectories $e^1_\pm(\varepsilon_1, \varepsilon_2)$, $e^2_\pm(\varepsilon_1, \varepsilon_2)$ for the potential W_2, for ε_1 fixed and equal to $\varepsilon_1^* = \sqrt{\frac{\sqrt{6}-1}{2}} - \frac{\sqrt{6}-1}{2}$ and $\varepsilon_2 = (\sigma(\varepsilon_1^*), +\infty)$. It can be seen that the inner region approaches a limiting shape as $\varepsilon_2 \to 0$ [14]. On the right are the level sets of $W_2(z)$ for $\varepsilon_1 = \varepsilon_1^*$ and $\varepsilon_2 = \sigma(\varepsilon_1^*)$ [14].

W_2 has global minima at $a^\pm = (\pm 1, 0)$ and obviously satisfies (h$_2$). Applying the theory in [3], we get that for $\varepsilon_1 > 0$ there exist precisely five connecting orbits between a^+ and a^-, which we denote by $e^1_\pm(\varepsilon_1, \varepsilon_2)$, $e^2_\pm(\varepsilon_1, \varepsilon_2)$, and $e_0(\varepsilon_1, \varepsilon_2)$. We denote by e_0 the 'scalar' connection mentioned in (h$_4$) that lies on the u_1-axis while the rest of the connections are symmetric in pairs with respect to the reflection $u_2 \mapsto -u_2$ (see Figure 3a) and are determined by the equation

$$u_2 - \frac{(\varepsilon_1^2 + 1)^2}{4\varepsilon_1(\varepsilon_2 - \varepsilon_1^2)} \ln \left(\frac{(\varepsilon_1 - u_2)^2 + u_1^2}{(u_2 + \varepsilon_1)^2 + u_1^2} \right) + \frac{(\varepsilon_1^2 + 1)^2}{4\varepsilon_2(\varepsilon_2 - \varepsilon_1^2)} \ln \left(\frac{(\varepsilon_2 - u_2)^2 + u_1^2}{(u_2 + \varepsilon_2)^2 + u_1^2} \right) = 0.$$

In addition, by applying [3], the action of each orbit can be calculated explicitly.

$$E_0 := E(e_0) = \frac{1}{\sqrt{2}} \left| 2 - \frac{(\varepsilon_1^2 + 1)^2}{\varepsilon_1(\varepsilon_2^2 - \varepsilon_1^2)} \arctan \varepsilon_1 + \right.$$
$$\left. + \frac{(\varepsilon_2^2 + 1)^2}{\varepsilon_2(\varepsilon_2^2 - \varepsilon_1^2)} \arctan \varepsilon_2 - \frac{(\varepsilon_2^2 + 1)^2 \pi}{2\varepsilon_2(\varepsilon_2^2 - \varepsilon_1^2)} + \frac{(\varepsilon_1^2 + 1)^2 \pi}{2\varepsilon_1(\varepsilon_2^2 - \varepsilon_1^2)} \right|,$$

$$E_I := E(e^1_\pm) = \frac{1}{\sqrt{2}} \left| 2 - \frac{(\varepsilon_2^2 + 1)^2}{\varepsilon_2(\varepsilon_2^2 - \varepsilon_1^2)} \arctan \varepsilon_2 + \frac{(\varepsilon_1^2 + 1)^2}{\varepsilon_1(\varepsilon_2^2 - \varepsilon_1^2)} \arctan \varepsilon_1 + \frac{(\varepsilon_2^2 + 1)^2 \pi}{2\varepsilon_2(\varepsilon_2^2 - \varepsilon_1^2)} \right|,$$

$$E_{II} := E(e^2_\pm) = \frac{1}{\sqrt{2}} \left| 2 + \frac{(\varepsilon_2^2 + 1)^2}{\varepsilon_2(\varepsilon_2^2 - \varepsilon_1^2)} \arctan \varepsilon_2 - \frac{(\varepsilon_1^2 + 1)^2}{\varepsilon_1(\varepsilon_2^2 - \varepsilon_1^2)} \arctan \varepsilon_1 \right|.$$

We observe that $e^2_- \cup e^2_+$ form the boundary of a region which increases unboundedly as $\varepsilon_2 \to \infty$ but approaches a limiting region as $\varepsilon_2 \to 0$, always enclosing all poles $(0, \pm\varepsilon_k i)$, $k = 1, 2$. On the other hand, $e^1_+ \cup e^1_-$ form the boundary of an interior region, containing only one pair of poles, that approaches limiting regions as $\varepsilon_2 \to 0$, $\varepsilon_2 \to \infty$.

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We note that

$$
E_0(\varepsilon_1, \varepsilon_2) \to
\begin{cases}
\infty, \text{ as } \varepsilon_1 \to 0, \\
\text{finite limits as } \varepsilon_1, \varepsilon_2 \to \varepsilon^* \neq 0 \\
\text{and also as } \varepsilon_1 \to \infty \text{ or } \varepsilon_2 \to \infty.
\end{cases}
$$

$$
E_{\mathrm{I}}(\varepsilon_1, \varepsilon_2) \to
\begin{cases}
\infty, \text{ as } \varepsilon_2 - \varepsilon_1 \to 0 \\
\text{finite limits, as } \varepsilon_1 \to 0, \ \varepsilon_2 \to \infty.
\end{cases}
$$

$$
E_{\mathrm{II}}(\varepsilon_1, \varepsilon_2) \to
\begin{cases}
\text{finite limit, as } \varepsilon_2 \to 0, \\
\infty, \text{ as } \varepsilon_2 \to \infty.
\end{cases}
$$

From the previous relations, we get that

$$
E_{\mathrm{I}} > E_{\mathrm{II}}, \text{ as } \varepsilon_2 \to 0 \text{ (with } \varepsilon_1 \text{ held constant)},
$$

$$
E_{\mathrm{I}} < E_{\mathrm{II}}, \text{ as } \varepsilon_2 \to \infty \text{ (with } \varepsilon_1 \text{ held constant)}.
$$

It then follows easily that there exists a continuous function $\varepsilon_1 \mapsto \sigma^*(\varepsilon_1)$ and $\varepsilon_1^* > 0$ such that

$$
(2.3) \qquad E_{\mathrm{II}}\big(\varepsilon_1, \sigma^*(\varepsilon_1)\big) = E_{\mathrm{I}}\big(\varepsilon_1, \sigma^*(\varepsilon_1)\big) < E_0\big(\varepsilon_1, \sigma^*(\varepsilon_1)\big),
$$

for $0 \leq \varepsilon_1 < \varepsilon_1^*$.

Theorem 7.1 applies to C^∞ mollifications of W_2 with $\varepsilon_2 = \sigma^*(\varepsilon_1)$, $0 \leq \varepsilon_1 < \varepsilon_1^*$, and produces an $\mathcal{H}_2^2$ equivariant solution, apparently not unique (in contrast to the previous example), which has the property that for some sequence $x_2^n \to \infty$, as $n \to \infty$,

$$
u(x_1, x_2^n) \to e_+^i(x_1), \quad u(x_1, -x_2^n) \to e_-^i(x_1),
$$

where $i \in \{1, 2\}$ but not known otherwise. We believe that for the example at hand it should be possible to prove that there exist two distinct solutions u^i satisfying

$$
\lim_{x_2 \to \pm\infty} u^i(x_1, x_2) = e_\pm^i(x_1), \text{ for } i = 1, 2,
$$

each mapping the plane $\mathbb{R}^2$ diffeomorphically to the corresponding region bounded by the connections. These are difficult questions; see [12] and the counterexample in [13] for related work.

Next we come to the higher-dimensional extensions that are studied in Section 8.

Extension 1. We consider the problem

$$
(2.4) \qquad
\begin{cases}
\Delta u - W_u(u) = 0, \text{ for } u : \mathbb{R}^3 \to \mathbb{R}^3, \\
\lim_{x_1 \to \pm\infty} u(x_1, x_2, x_3) = (\pm a, 0, 0).
\end{cases}
$$

with hypotheses

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(h_a) (Minima) The potential W is C^2, $W : \mathbb{R}^3 \to \mathbb{R}_+ \cap \{0\}$ with precisely two nondegenerate minima a^+ and a^-, where $a^+ = (a, 0, 0)$, $a^- = (-a, 0, 0)$, for $a > 0$, and $\partial^2 W(u) \geq c^2 \mathrm{Id}$, for $|u - a^+| < r_0$. Also, $W_u(u) \cdot u > 0$ for u outside a ball centered at the origin and containing the minima.

(h_b) (Symmetries) We assume that W is invariant under the symmetry group generated by the reflections with respect to the planes $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_2 = \pm u_3$. This group has eight elements and is isomorphic to the dihedral group $\mathcal{H}_2^4$ in the (u_2, u_3)-plane (symmetries of the square). We denote it by $\mathcal{H}_3^4$.

We thus assume
$$W(gu) = W(u), \text{ for } g \in \mathcal{H}_3^4$$

(h_c) (Connections) We assume, besides e_0, the existence of precisely one pair of connecting orbits between a^+ and a^- on each of the planes $u_3 = 0$, $u_2 = 0$, $u_2 = u_3$, $u_2 = -u_3$, denoted by e_+^1, e_-^1, e_+^2, e_-^2, e_+^3, e_-^3, and e_+^4, e_-^4 respectively. Symmetry implies
$$E_{\mathrm{I}} := E(e_\pm^1) = E(e_\pm^2),$$
$$E_{\mathrm{II}} := E(e_\pm^3) = E(e_\pm^4).$$

Consequently, in total we have nine connecting orbits between a^+ and a^- (see Figure 4). We assume that

(2.5) $$E(e_0) > E_{\mathrm{I}} > E_{\mathrm{II}}$$

Thus,

- E_{II} is the minimum value of the action among all curves connecting a^+ and a^-.
- E_{I} is the minimum value of the action among curves connecting a^+ and a^-, and lying entirely on either of the planes $u_3 = 0$, $u_2 = 0$.

(h_d) (Q-monotonicity) Let $D = \{(u_1, u_2, u_3) \mid u_1 \geq 0\}$. We assume that there exists a C^2 convex function $Q : D \setminus \{a^+\} \to \mathbb{R}_+ \cup \{0\}$ with $Q(u) > 0$ for $u \neq a^+$ and $Q(u) = |u - a^+|$ for $|u - a^+| < r_0$, satisfying the relation
$$W_u(u) \cdot Q_u(u) \geq 0, \text{ for } u \in D.$$

We study the problem in the class of $\mathcal{H}_3^2$-equivariant vector fields. Note that we impose more symmetry on the potential than required for the solution. Under the hypotheses (h_a)–(h_d), we establish in Theorem 8.1 the existence of a genuine three-dimensional, $\mathcal{H}_3^2$-equivariant, bounded solution to (2.4). In fact, we exhibit in this example a full hierarchy of solutions: a three-dimensional solution, which approaches in the x_1-direction the critical points $a^\pm$ (zero-dimensional solutions), while in the x_2- and x_3-directions, approaches distinct two-dimensional solutions, which themselves have boundaries made up of one-dimensional connections. The ideas are as follows. The existence of the nonequivariant connecting orbits $e_\pm$ in Example 1 above, with action less than that of e_0, exclude e_0 as a candidate for a (trivial) two-dimensional minimizer and imply the existence of a genuine two-dimensional solution. In the
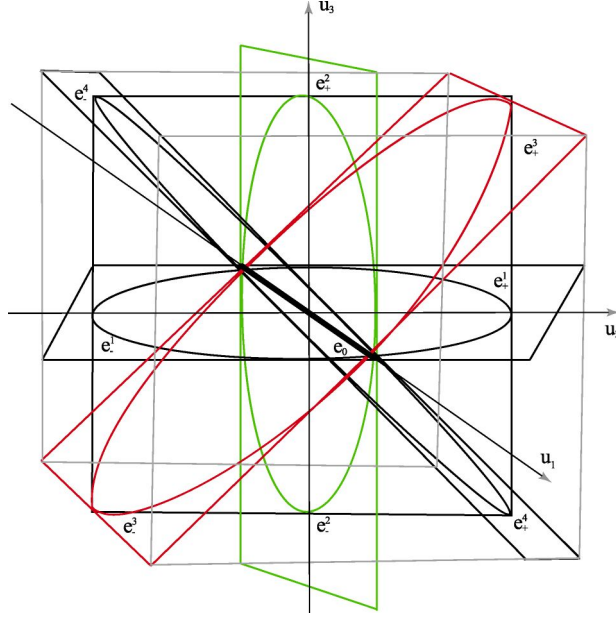
Figure 4: The connecting orbits between a^+ and a^-.

present example, in addition to e_0, we have two two-dimensional $\mathcal{H}_3^2$-equivariant solutions. The $\mathcal{H}_3^4$ symmetry together with the hypothesis $E_{\mathrm{I}} > E_{\mathrm{II}}$ implies the existence of non-$\mathcal{H}_3^2$-equivariant two-dimensional solutions which play an role analogous to $e_\pm$ in the exclusion of the $\mathcal{H}_3^2$-equivariant two-dimensional solutions as candidates for (trivial) three-dimensional minimizers and thus implying the existence of an $\mathcal{H}_3^2$-equivariant genuine three-dimensional solution.

Extension 2. We conclude Section 2 with an n-dimensional extension of Theorem 1.1 which we now describe. Let $u = (u_1, \ldots, u_n)$ be the typical element of $\mathbb{R}^n$; we write $u = (u_1, u')$ with $u' = (u_2, \ldots, u_n) \in \mathbb{R}^{n-1}$. We denote by $T_j : \mathbb{R}^n \to \mathbb{R}^n$ the reflection with respect to the plane $u_j = 0$, that is, the map defined by

$$(2.6) \qquad\qquad T_j(u_1, \ldots, u_j, \ldots, u_n) := (u_1, \ldots, -u_j, \ldots, u_n).$$

We let G be the reflection group generated by T_j, $j = 1, \ldots, n$, and G' the reflection group generated by T_j, $j = 2, \ldots, n$. If G is a group, we denote by $|G|$ the order of G, that is, the number of elements of G. Note that $|G| = 2^n$, $|G'| = 2^{n-1}$. We denote by F, F', $F' \cap F \neq \varnothing$, convex (open) fundamental regions defined in $\mathbb{R}^n$ by the reflection groups G, G' (see Section 3).

In this case, the hypotheses are as follows.

(h$_\mathrm{i}$) (Minima) Assume $W : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function such that there exists an $\alpha > 0$ with $W(\pm\alpha, 0') = 0$, $W(u) > 0$, for all $u \neq (\pm\alpha, 0')$ and $(\pm\alpha, 0')$ are nondegenerate zeros of W, as in (h$_1$).

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(h_{ii}) (Symmetries) W is equivariant with respect to G, that is, $W(gu) = W(u)$, for all $g \in G$. Also we assume that $W(u) \geq \max_{\partial C_0}[W]$ for u outside a certain bounded, G-symmetric, convex set C_0, as in (h_2).

(h_{iii}) (Q-monotonicity) We assume that there exists a C^2 function $Q : D \setminus \{a^+\} \to \mathbb{R}_+ \cup \{0\}$, convex, with $Q(u) > 0$ for $u \in D$ and $Q(u) = |u - a^+|$ for $|u - a^+| < r_0$, satisfying the relation

$$W_u(u) \cdot Q_u(u) \geq 0, \text{ for } u \in D.$$

(h_{iv}) (Connections) For each $e \neq e_0 \in A \subset W^{1,2}(\mathbb{R})$, (the analogue of ($h_3$)), the set of minimizers of the connection problem, we have

(2.7)
$$\#\tilde{e} = |G'|, \text{ for } e \in A$$

where $\#\tilde{e}$ denotes the cardinality of the set $\tilde{e}$, $\tilde{e} := \{ge \mid g \in G'\}$.

Under these hypotheses, we establish in Theorem 8.3 the existence of a genuinely n-dimensional, G-equivariant solution to

(2.8)
$$\Delta u - W_u(u) = 0,$$

(2.9)
$$\lim_{x_1 \to \pm\infty} u(x_1, x') = (\pm\alpha, 0').$$

In the following, the square brackets on the side of sections, theorems etc. indicate which of the hypotheses are needed for the developments there.

3   The constrained problem [h_1]

Let $\Omega_{R,\mu} = \{(x_1, x_2) \mid |x_1| < \mu R, \ |x_2| < R\}$ and $C^+_{R,\mu,\eta} = \{(x_1, x_2) \in \Omega_{R,\mu} \mid \eta R \leq x_1 \leq \mu R\}$, where $R \in [1, \infty)$, $\mu \in [1, +\infty]$, $\frac{1}{2} < \eta < \mu$, and $C^-_{R,\mu,\eta} = \{(x_1, x_2) \in \Omega_{R,\mu} \mid -\mu R \leq x_1 \leq -\eta R\}$. Finally, the domain $\Omega_{R,\infty}$, for $\mu = \infty$, is the strip $|x_2| < R$. Consider the equivariant Sobolev space $W^{1,2}_E(\Omega_{R,\mu}) = \{u : \Omega_{R,\mu} \to \mathbb{R}^2 \mid u \in W^{1,2}(\Omega_{R,\mu}), u \ \mathcal{H}^2_2\text{-equivariant}\}$. We consider, for $r < r_0$ fixed, the set

(3.1)
$$U^c_{R,\mu} := \left\{ u \in W^{1,2}_E(\Omega_{R\mu}) \mid |u(x) - a^\pm| \leq r, \text{ a.e. } x \in C^\pm_{R,\mu,\eta} \right\}$$

and the functional

$$J_{R,\mu}(u) = \int_{\Omega_{R,\mu}} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dx$$

which we denote by J whenever there is no risk of confusion.

Proposition 3.1. *Let* $1 \leq R < \infty$, $1 \leq \mu \leq \infty$, $\frac{1}{2} < \eta \leq \mu$ *and* $r < r_0$ *fixed, where* r_0 *as in* (h_1). *Then, the problem*

(3.2)
$$\min_{U^c_{R,\mu}} \int_{\Omega_{R,\mu}} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dx$$

has a solution $u_{R,\mu} \in W^{1,2}_E(\Omega_{R,\mu})$ *for* $\mu < \infty$ *and* $u_{R,\infty} \in (W^{1,2}_E)_{loc}(\Omega_{R,\infty})$ *to*

(3.3)
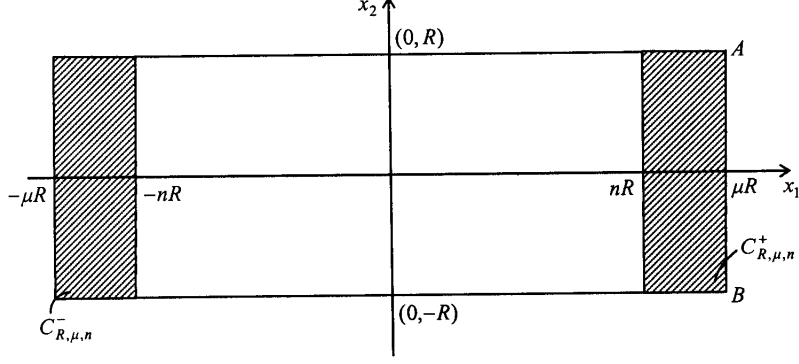$$J(u_{R,\mu}) = \inf_{U^c_{R,\mu}} [J]$$

Figure 5: The sets $C^+_{R,\mu,\eta}$ and $C^-_{R,\mu,\eta}$ [10].

Proof. For $\mu < \infty$, we fix R and μ, and define the linear function $u_{\text{aff}} : \Omega_{R,\mu} \to \mathbb{R}^2$, such that

$$(3.4) \qquad u_{\text{aff}}(x) := \begin{cases} a^-, & \text{for} \quad x_1 \in [-\mu R, -1], \\ \dfrac{1-x_1}{2} a^- + \dfrac{1+x_1}{2} a^+, & \text{for} \quad x_1 \in [-1,1], \\ a^+, & \text{for} \quad x_1 \in [1, \mu R]. \end{cases}$$

u_{aff} belongs to $U^c_{R,\mu}$ for every $R \geq 1$, $\mu \geq \eta$ and satisfies the estimate

$$(3.5) \qquad\qquad\qquad\qquad J(u_{\text{aff}}) \; < \; CR.$$

Since $W \geq 0$, it follows that $0 \leq \inf_{U^c_{R,\mu}} J < J(u_{\text{aff}}) < CR$ where without loss of generality we assumed the middle inequality to be strict. Let $\{u_n\}$ be a minimizing sequence of J, that is, $J(u_n) \to \inf_{U^c_{R,\mu}} [J]$. For the sequence $\{u_n\}$ we have the following estimates

$$(3.6) \qquad \begin{cases} \text{(i)} \quad \displaystyle\int_{\Omega_{R,\mu}} \frac{1}{2} |\nabla u_n|^2 \, \mathrm{d}x < J(u_{\text{aff}}) < CR, \\[2mm] \text{(ii)} \quad \displaystyle\int_{\Omega_{R,\mu}} |u_n|^2 \, \mathrm{d}x < C(R,\mu). \end{cases}$$

where in (3.6)(ii), $C(R,\mu)$ denotes a constant depending on R, μ. Then, there exists a subsequence, by weak compactness, which we still denote by $\{u_n\}$, such that

$$u_n \rightharpoonup u, \;\; \text{weakly in } W^{1,2}_{\text{E}}(\Omega_{R,\mu}).$$

By lower semicontinuity in $L^2_{\text{E}}(\Omega_{R,\mu})$, it follows that

$$(3.7) \qquad\qquad\qquad \liminf_{n\to\infty} \int_{\Omega_{R,\mu}} |\nabla u_n|^2 \, \mathrm{d}x \geq \int_{\Omega_{R,\mu}} |\nabla u|^2 \, \mathrm{d}x$$

and by the compactness of the embedding $W^{1,2}_{\text{E}}(\Omega_{R,\mu}) \subset\subset L^2_{\text{E}}(\Omega_{R,\mu})$ and the lemma of Fatou we have

$$(3.8) \qquad\qquad\qquad \liminf_{n\to\infty} \int_{\Omega_{R,\mu}} W(u_n) \, \mathrm{d}x \geq \int_{\Omega_{R,\mu}} W(u) \, \mathrm{d}x,$$

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and thus the proposition holds.

For $\mu = \infty$, let now $\Omega_{R,\infty}$ stand for the infinite strip $\{(x_1, x_2) | |x_2| < R\}$. We consider the Fréchet space[3] $W^{1,2}_{\text{loc}}(\Omega_{R,\infty})$ whose topology is defined by the seminorms of $W^{1,2}_m :=$ $W^{1,2}([-m, m] \times [-R, R])$. We assume now that $\{u_n\}$ is bounded in the locally convex sense in $W^{1,2}_{\text{loc}}(\Omega_{R,\infty})$. Utilizing a standard diagonal argument and the representation of functionals in $(W^{1,2}_m)^*$ induced by the embedding $W^{1,2}_m \hookrightarrow L^2_m \times (L^2_m)^2$, it follows that $\{u_n\}$ is weakly precompact in the topology of $W^{1,2}_{\text{loc}}(\Omega_{R,\infty})$ generated by the duals $(W^{1,2}_m)^*$, that is, there exist a subsequence $\{u_{n_k}\}$ denoted again by $\{u_n\}$, and a u in $W^{1,2}_{\text{loc}}(\Omega_{R,\infty})$, such that, for every $m = 1, 2, \ldots$

$$u_n \rightharpoonup u \text{ weakly in } W^{1,2}_m, \text{ as } n \to \infty.$$

By the estimate (3.6)(i) and the local version of (3.6)(ii), we may assume that, up to a further subsequence, $u_n \to u$ a.e. on $\Omega_{R,\mu}$, due to the embedding $W^{1,2}(\Omega_{R,\mu}) \subset\subset L^2(\Omega_{R,\mu})$. This validates (3.7) and (3.8) above and finishes the proof. $\qquad\square$

4   The positivity property

Let V be a real Euclidean vector space, and let $O(V)$ stand for the orthogonal group. For every finite subgroup G of $O(V)$ a *fundamental region* is defined as a set F with the following properties.

(i) F is open in V,

(ii) $F \cap TF = \varnothing$ if $I \neq T \in G$,

(iii) $V = \cup\{(TF)^- \mid T \in G\}$,

where with $(\ )^-$ we denote the closure of the set. The fundamental region F can be chosen to be convex, actually a simplex (see [10]). More generally, if X is a subset of V, invariant under G, then a subset D is a *fundamental domain* if it is of the form

$$D = X \cap F.$$

If $G = \mathcal{H}^2_2$, a fundamental region is $F = \{(u_1, u_2) \mid u_1 \geq 0, u_2 \geq 0\}$. For $X = \Omega_{R,\mu}$, we take as fundamental domain the set $\Omega^1_{R,\mu} = \Omega_{R,\mu} \cap F$.

Proposition 4.1. [**h₂**] *Let $u_{R,\mu}$, for $R, \mu \in [1, \infty)$, be the minimizing function of the constrained problem (3.2). Then, there exist $u^*_{R,\mu} \in U^c_{R,\mu}$ with the properties*

(4.1)
$$\begin{cases} J(u^*_{R,\mu}) \leq J(u_{R,\mu}) \\ u^*_{R,\mu}(\Omega^1_{R,\mu}) \subset F^- \end{cases}$$

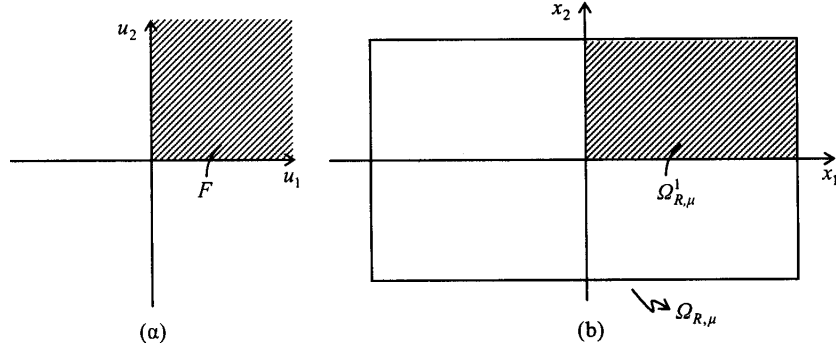[3]We owe this argument to N. Katzourakis

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Figure 6: The fundamental region F of $\mathcal{H}_2^2$ and the fundamental domain $\Omega_{R,\mu}^1 = \Omega_{R,\mu} \cap F$ [10].

Proof. Set

$$(4.2) \qquad \Lambda u := \begin{cases} u, & u \in F \\ T_1^{-1} u, & u \in T_1(F) \\ (T_2 T_1)^{-1} u, & u \in T_2 T_1(F) = S(F) \\ T_2^{-1} u, & u \in T_2(F) \end{cases}$$

Clearly, Λ maps $\mathbb{R}^2$ into F. Also, it can be checked that

$$(4.3) \qquad |\Lambda(u_A) - \Lambda(u_B)| \leq |u_A - u_B|,$$

where $|\cdot|$ is the Euclidean norm.

Next, we define the operator

$$(4.4) \qquad (Lu)(x) := \Lambda u(x), \text{ for } x \in \Omega_{R,\mu}^1,$$

and extend by equivariance on $\Omega_{R,\mu}$. We will show that

$$(4.5) \qquad L : U_{R,\mu}^c \to U_{R,\mu}^c,$$

which means that L preserves Sobolev equivariance and the constraint.

We begin by verifying that L preserves Sobolev equivariance. By standard approximation arguments, the only source of difficulty is the possible loss of continuity along the symmetry lines where the gluing in the definition of L takes place. We check two cases and leave the rest to the reader.

We consider x^+, $\bar{x}$, x^- as in Figure 7 with $T_1 x^+ = x^-$ and $|x^+ - x^-|$ small, and $T_1 \bar{x} = \bar{x}$. We would like to show that $|(Lu)(x^+) - (Lu)(x^-)|$ is small for $|u(x^+) - u(x^-)|$ small. By equivariance, $T_1(u(\bar{x})) = u(T_1 \bar{x}) = u(\bar{x})$ and therefore, $u(\bar{x})$ lies on the u_2-axis. We assume
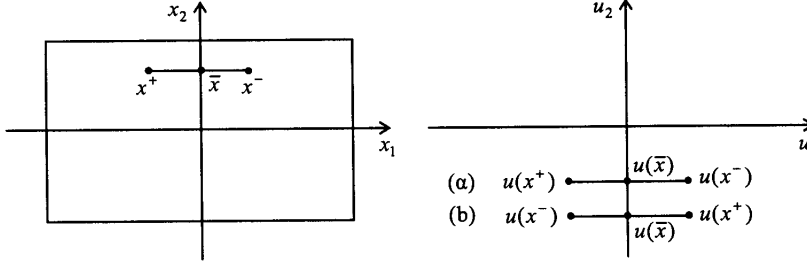
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Figure 7: The points x^+, $\bar{x}$, x^-, and the corresponding $u(x^-)$, $u(\bar{x})$, $u(x^+)$ [10].

that $u(x^-)$, $u(\bar{x})$, $u(x^+)$ are as in Figure 7. Then,

(i) $Lu(x^-) = \Lambda u(x^-) = T_2 u(x^-)$,

   $Lu(x^+) = T_1 \Lambda u(T_1^{-1} x^+) = T_1 \Lambda u(x^-) = T_1 T_2 u(x^-) = T_2 T_1 u(x^-) = T_2 u(x^+)$,

(ii) $Lu(x^-) = \Lambda u(x^-) = T_1 T_2 u(x^-) = T_2 T_1 u(x^-) = T_2 u(x^+)$,

   $Lu(x^+) = T_1 \Lambda u(T_1^{-1} x^+) = T_1 \Lambda u(x^-) = T_1 T_1 T_2 u(x^-) = T_2 u(x^-)$.

consequently, continuity is verified in these cases. The verification of the constraint is straightforward. Finally, we define

$$u^*_{R,\mu} := Lu_{R,\mu}$$

and verify that $u^*_{R,\mu}$ does not increase the functional J. Indeed,

$$W\big((Lu)(x)\big) = W\big(g\Lambda u(g^{-1}x)\big) = W\big(\Lambda u(g^{-1}x)\big) = W\big(u(g^{-1}x)\big)$$

and consequently, the term W of the functional J does not change since T_i is an isometry. On the other hand, the term $\int_{\Omega_{R,\mu}} |\nabla u|^2 \, dx$ does not increase by (4.3). $\qquad\square$

Corollary 4.1. [$\mathbf{h_1}$, $\mathbf{h_2}$] *Without loss of generality, we may assume that the minimizer $u_{R,\mu}$ of the constrained problem satisfies*

(4.6) $$u_{R,\mu}(\Omega^1_{R,\mu}) \subseteq F^-$$

Next we need an *a priori* bound.

Lemma 4.1. *There is an $M > 0$, independent of R, μ, η, such that*

$$|u_{R,\mu}(x)| < M, \ \text{for } x \in \Omega_{R,\mu}.$$

Proof. For the convex set C_0 introduced in (h_1), we consider the mapping $\Lambda : \mathbb{R}^2 \to C_0$,

(4.7) $$\Lambda u := \begin{cases} Pu, & \text{if } u \notin C_0, \\ u, & \text{if } u \in C_0, \end{cases}$$

where Pu is the projection of u on ∂C_0. By (h_1), $W(\Lambda u) \leq W(u)$. Also, the mapping Λ is nonexpansive in the Euclidean norm. We set $(Lu)(x) := \Lambda u(x)$ and notice that L preserves equivariance, honors the constraint, and reduces $J_{R,\mu}$. It follows that the minimizer $u_{R,\mu}$ of the constrained problem takes values in C_0. Thus (4.7) holds. $\qquad\square$

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5 Removing the constraint—the case of the bounded domain

Given $u : x \in \mathbb{R}^2 \to \mathbb{R}^2$, we write $u(x) - a^{\pm}$ in polar form,

$$u(x) - a^{\pm} = |u(x) - a^{\pm}| \frac{u(x) - a^{\pm}}{|u(x) - a^{\pm}|} = \rho^{\pm}(x) n^{\pm}(x),$$

with $\rho^{\pm} : x \in \mathbb{R}^2 \to \mathbb{R}_+$ and $n^{\pm} : x \in \mathbb{R}^2 \to \mathbb{S}^1$. So, if $u \in U^c_{R,\mu}$, we have

$$u(x) = a^+ + \rho^+(x) n^+(x), \text{ with } \rho^+(x) \leq r \text{ for } x \in C^+_{R,\mu,\eta},$$

and similarly for $x \in C^-_{R,\mu,\eta}$. We notice that the polar form is well defined for $\rho(x) \neq 0$.

For $u \in W^{1,2}_{\text{loc}}$, it follows that $\rho, n \in W^{1,2}_{\text{loc}}$ and moreover, $|\nabla u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla n|^2$. On the other hand, on the set $\{u = a\}$, we have $|\nabla u| = 0$ a.e. Therefore, for any measurable set S, we have

$$\int_S |\nabla u|^2 \, dx = \int_{S \cap \{\rho > 0\}} \{|\nabla \rho(x)|^2 + \rho^2(x) |\nabla n(x)|^2\} \, dx.$$

Lemma 5.1. [$\mathbf{h_1}$] *Suppose $u_{R,\mu}$ is a minimizer of the constrained problem (3.2). Then, the following estimate holds.*

$$(5.1) \qquad \rho^+_{R,\mu}(x) \; \leq \; r \, \frac{\cosh\big(c(R\mu - x_1)\big)}{\cosh\big(c(\mu - \eta)R\big)}, \quad a.e. \; x \in C^+_{R,\mu,\eta}.$$

where c as in (h_1)*, with an analogous estimate for $x \in C^-_{R,\mu,\eta}$. Here, $1 \leq R < \infty$, $1 \leq \mu \leq \infty$, and $\frac{1}{2} < \eta < \mu$, for $r < r_0$.*

Note that an important consequence of the corollary is that the constraint is realized, if at all, on the line $\{x_1 = \eta R\}$.

Proof. Suppose that

$$(5.2) \qquad \begin{cases} \Delta w - c^2 w \geq 0, \\ Bw \leq 0, \end{cases}$$

weakly in the space $W^{1,2}_\#(C^+_{R,\mu,\eta})$, the latter defined as the completion in the $W^{1,2}$ norm of the space

$$\{f \in C^\infty(\overline{C^+_{R,\mu,\eta}}) \cap W^{1,2}(C^+_{R,\mu,\eta}) \mid f^+ = 0 \text{ on } \{x_1 = \eta R\}\},$$

where

$$Bw := \begin{cases} w, & \text{on } x_1 = \eta R, \\ \dfrac{\partial w}{\partial n}, & \text{on } \partial_L C^+_{R,\mu,\eta}\big(:= \partial C^+_{R,\mu,\eta} \setminus \{x_1 = \eta R\}\big), \end{cases}$$

and (5.2) is meant in the sense

$$(5.3) \qquad \int_{C_{R,\mu,\eta}} \{\nabla w \nabla \phi + c^2 w \phi\} dx \leq 0,$$

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for $w, \phi \in W_{\#}^{1,2}(C_{R,\mu,\eta}^+)$ with $\phi \geq 0$ a.e. Then, we claim that

(5.4)
$$w \leq 0, \text{ a.e. in } C_{R,\mu,\eta}^+.$$

To prove the claim, by density we can take $\phi := w^+$ in (5.3) and so we can conclude that

$$0 \geq \int_{C_{R,\mu,\eta}^+} \left\{ \nabla w \nabla w^+ + c^2 w w^+ \right\} \mathrm{d}x = \int_{C_{R,\mu,\eta}^+} \left\{ |\nabla w^+|^2 + c^2 |w^+|^2 \right\} \mathrm{d}x = 0,$$

thus, $w^+ = 0$ in $C_{R,\mu,\eta}^+$. Next we will show that

(5.5)
$$\Delta \rho_{R,\mu} \geq \rho_{R,\mu} c^2 \text{ weakly in } W^{1,2}(C_{R,\mu,\eta}^+).$$

For showing (5.5), we consider $u_\varepsilon(x) = u_{R,\mu}(x) + \varepsilon \hat{p}(x) n(x)$ with $\hat{p}(x) \leq 0$ in $C_{R,\mu,\eta}^+$, $\hat{p} \in C_0^\infty(C_{R,\mu,\eta}^+)$. We notice that $|u_\varepsilon(x) - a^\pm| = |\rho_{R,\mu}(x) + \varepsilon \hat{p}(x)| \leq r$ in $C_{R,\mu,\eta}^\pm$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} J(u_\varepsilon) \geq 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \int_{\Omega_{R,\mu,\eta}} \left\{ \frac{1}{2} |\nabla u_\varepsilon|^2 + W(u_\varepsilon) \right\} \mathrm{d}x \geq 0$$

$$\Leftrightarrow \quad \int_{C_{R,\mu,\eta}} \left\{ \nabla \rho_{R,\mu} \nabla \hat{p} + (\rho_{R,\mu}\hat{p})|\nabla n(x)|^2 + \hat{p} W_u(u_{R,\mu}) n(x) \right\} \mathrm{d}x \geq 0$$

from which it follows that

$$\int_{C_{R,\mu}} \left\{ \nabla \rho_{R,\mu} \nabla \hat{p} + \hat{p} W_u(u_{R,\mu}) n(x) \right\} \mathrm{d}x \geq 0.$$

Utilizing (h$_1$), we obtain from this

$$\int_{C_{R,\mu}} \left\{ \nabla \rho_{R,\mu} \nabla \hat{p} + c^2 \hat{p} \, \rho_{R,\mu} \right\} \mathrm{d}x \geq 0,$$

and therefore (5.5) has been established.

Next we will show that $\rho_{R,\mu} < r$ a.e. in the interior of $C_{R,\mu,\eta}$ from which it will follow, up to a modification on a null set, that $u_{R,\mu}$ is a classical solution of

(5.6)
$$\Delta u_{R,\mu} - W_u(u_{R,\mu}) = 0, \text{ in the interior of } C_{R,\mu,\eta}.$$

Suppose now for the sake of contradiction that $\rho_{R,\mu} = r$ on a set A of positive measure. However, this is in conflict with $\Delta \rho_{R,\mu} \geq c^2 \rho_{R,\mu}$ in $W^{1,2}(C_{R,\mu})$ since $\nabla \rho_{R,\mu} = 0$ a.e. on this set A. Therefore, $\rho_{R,\mu}(x) < r$ a.e. in $C_{R,\mu,\eta}^+$ as required.

In the following we show that

(5.7)
$$\frac{\partial \rho_{R,\eta}}{\partial n} = 0 \text{ on } \partial_L C_{R,\mu,\eta} \setminus \{A, B\},$$

where A, B are the corners. For x^* in a subset of points $\partial_L C_{R,\mu} \setminus \{A, B\}$ such that $\rho_{R,\mu}(x^*) < r$ a.e. on it, the natural boundary conditions hold classically and so (5.7) is valid. Therefore, the case of interest is when $\rho_{R,\mu}(x^*) = r$. We notice that in the interior of $C_{R,\mu,\eta}$, (5.6) is satisfied

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classically and that $u_{R,\mu}$ is regular. From the bound $|u_{R,\mu}| < \text{const}$, which holds uniformly in the interior of $C_{R,\mu,\eta}$, we obtain by elliptic regularity that $|\nabla\rho_{R,\mu}| < \text{const}$, on the boundary with a similar estimate on the second-order derivatives. Consequently, $\rho_{R,\mu}(x)$ is continuous at x^* and the outer normal derivative $\partial\rho_{R,\mu}/\partial n$ exists at x^*. We know that $\Delta\rho_{R,\mu} \geq c^2\rho_{R,\mu}$ classically in the interior of $C_{R,\mu}$ and by the preceding argument, $\rho_{R,\mu}$ is continuous at $x = x^*$ and $\partial\rho_{R,\mu}/\partial n\,(x^*)$ exists. Applying the Hopf lemma, we obtain

$$(5.8) \qquad\qquad \frac{\partial\rho_{R,\mu}}{\partial n}(x^*) > 0.$$

We now set $u_\varepsilon(x) = u_{R,\mu} + \varepsilon\hat{p}(x)n$, $\hat{p} \leq 0$ smooth with $\text{supp}(\hat{p}) \subseteq B(x^*;\delta) \cap \overline{C_{R,\mu,\eta}}$, $0 < \delta \ll 1$. Then, $u_\varepsilon \in U^c_{R,\mu}$ and

$$(5.9) \qquad 0 \leq \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \int_{\Omega_{R,\mu}} \left\{\frac{1}{2}|\nabla u_\varepsilon|^2 + W(u_\varepsilon)\right\}^2 \mathrm{d}x \overset{(5.6)}{=} \int_{\partial\Omega_{R,\mu}} \frac{\partial\rho_{R,\mu}}{\partial n}\hat{p}\,\mathrm{d}S,$$

which however is in contradiction to (5.8). Therefore, $\rho_{R,\mu}(x^*) = r$ cannot possibly hold and so (5.7) is valid.

To conclude, we set

$$v := \rho^+_{R,\mu}(x) - r\frac{\cosh\big(c(R\mu - x_1)\big)}{\cosh\big(c(\mu - \eta)R\big)}.$$

We will show that v satisfies (5.2). By the preceding argument, it follows that $\Delta v - c^2 v \geq 0$ classically in the interior of $C^+_{R,\mu,\eta}$. Thus, given ϕ as in the definition of (5.5), we have

$$0 \leq \int_{C^+_{R,\mu}} \left\{\Delta v - c^2 v\right\}\phi\,\mathrm{d}x = \int_{C^+_{R,\mu}} \left\{-\nabla v\nabla\phi - c^2 v\phi\right\}\mathrm{d}x + \int_{\partial_L C_{R,\mu,\eta}} \frac{\partial v}{\partial n}\phi\,\mathrm{d}S$$

$$\overset{(5.7)}{=} \int_{C^+_{R,\mu}} \left\{-\nabla v\nabla\phi - c^2 v\phi\right\}\mathrm{d}x.$$

Finally we note that the points A, B are negligible in the boundary integral since $\big|\nabla v\big| < \text{const}$, up to the boundary. The proof of Lemma 5.1 is complete. $\qquad\square$

Corollary 5.1. [$\mathbf{h_1}$] *Suppose $u_{R,\mu}$ is the minimizer of the constrained problem (3.2). Then the following estimate is true.*

$$(5.10) \qquad\qquad \rho^+_{R,\mu}(x) \leq 2re^{-c(x_1 - \eta R)}, \quad a.e. \ x \in C^+_{R,\mu,\eta},$$

where $1 \leq R < \infty$, $\frac{1}{2} < \eta < \mu$, and fixed $r < r_0$, with an analogous result for $x \in C^-_{R,\mu,\eta}$.

We begin with a result that is of some independent interest, to be utilized later.

Lemma 5.2 (Chop-off Lemma, cf. [5]; requires only the monotonicity of $\lambda \mapsto W(a + \lambda w)$, $|w| = 1$, $\lambda < r_0$.). *Suppose A is an open set in $\mathbb{R}^2$ and u a global minimizer of*

$$\min_{\rho^+|_\Gamma = r} \int_A \left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}\mathrm{d}x < \infty$$

subject to a unilateral constraint $|\rho^+| \le M$ on an open subset $A' \subseteq A$. We employ the representation $u(x) = a^+ + \rho^+(x)n^+(x)$, $2r < r_0$, and $\Gamma \subset \partial A$ and possibly empty. Then the following estimate holds.

$$(5.11) \qquad\qquad\qquad \rho^+(x) \le r \text{ a.e. on } A.$$

Proof. We first establish the estimate

$$(5.12) \qquad\qquad\qquad \rho^+(x) \le 2r, \text{ a.e. on } A.$$

We begin by proving it under the hypothesis $\rho^+(x) > r$ on A. For $r_0 > \hat{r} > 2r$, we consider the sets

$$A^+ = \{x \in A \mid \rho^+(x) > \hat{r}\}, \quad A^- = \{x \in A \mid \rho^+(x) \le \hat{r}\},$$

and define the function

$$\theta(\tau) := \begin{cases} 1, & \tau \le r, \\ \dfrac{\hat{r} - \tau}{\hat{r} - r}, & r < \tau < \hat{r}, \\ 0, & \tau \ge \hat{r}. \end{cases}$$

We also define $\hat{u}(x) := a + r\theta(\rho^+(x))n^+(x)$ and $|\hat{u}(x) - a| = r\theta(\rho^+(x)) =: \hat{p}(x)$. We will establish the validity of the following estimate.

$$|\nabla \hat{u}(x)| \le |\nabla u(x)|.$$

We recall that $|\nabla u|^2 = |\nabla \rho^+|^2 + (\rho^+)^2 |\nabla n^+|^2$, thus,

$$|\nabla \hat{u}(x)|^2 = |\nabla \hat{p}(x)|^2 + \hat{p}^2(x) |\nabla n^+|^2,$$

with

$$|\nabla \hat{p}(x)|^2 = r^2 |\nabla_x \theta(\rho^+(x))|^2 = r^2 (\theta')^2 |\nabla \rho^+(x)|^2 \le \frac{r^2}{(\hat{r} - r)^2} |\nabla \rho^+(x)|^2 \le |\nabla \rho^+(x)|^2,$$

where we have used the assumption $\hat{r} > 2r$. Thus, in A and for $\hat{r} > 2r$ we have

$$|\nabla \hat{u}(x)|^2 = |\nabla \hat{p}(x)|^2 + \hat{p}^2 |\nabla n|^2 \le |\nabla \rho(x)|^2 + r^2 |\nabla n|^2 \le |\nabla \rho(x)|^2 + \rho^2(x) |\nabla \eta|^2 = |\nabla u(x)|^2.$$

In A, the inequalities $|\hat{u}(x) - a| \le r \le \rho(x)$ hold. This and the monotonicity assumption on W imply

$$\int_A W(\hat{u}(x)) dx \le \int_A W(u(x)) dx$$

which is strict unless $|A^+| = 0$. The conclusion now follows from

$$\int_A \left\{ \frac{1}{2} |\nabla \hat{u}|^2 + W(\hat{u}) \right\} dx < \int_A \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dx$$

and the fact that the transformation does not affect the unilateral constraint.

We now eliminate the hypothesis $\rho^+(x) > r$ on A. We set $B = \{x \in A | \rho^+(x) > r\}$. If $B = \varnothing$ we are done; if $B \neq \varnothing$, then B is open in which case we can repeat the argument above by replacing A with B. The proof of (5.12) is complete.

Now, define

$$w(x) = a^+ + \min\{\rho^+, r\}n(x) =: a^+ + \tilde{\rho}(x)n(x)$$

and notice that $|\nabla\tilde{\rho}| \leq |\nabla\rho^+|$. By (5.12) and the monotonicity assumption on W, we have that

$$\int_A \left\{\frac{1}{2}|\nabla w|^2 + W(w)\right\} \mathrm{d}x < \int_A \left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\} \mathrm{d}x$$

which is strict unless $\rho^+(x) \leq r$ a.e. This completes the proof of the lemma. $\qquad\square$

Now we are ready to establish the main result of this section.

Theorem 5.1. *We assume that W satisfies hypotheses (h_1), (h_2). Then, we can determine $R_0 \geq 1$ and $\mu_0 \geq 1$, so that for $R > R_0$ and $\mu \geq \mu_0$, there exists a minimizer $u_{R,\mu}$ of the constrained problem which does not realize the constraint and thus satisfies, weakly in $W^{1,2}(\Omega_{R,\mu})$, the problem*

(5.13)
$$\begin{cases} \Delta u - W_u(u) = 0, & \text{in } \Omega_{R,\mu}, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega_{R,\mu}. \end{cases}$$

Moreover,

(5.14)
$$u(x) \not\equiv 0$$

and

(5.15)
$$u(x) \not\equiv e_0(x_1), \ (e_0(0) = 0).$$

Before we present the proof, we will make two remarks. First, the weak formulation of (5.13) is

$$\int_{\Omega_{R,\mu}} \left\{\nabla u_{R,\mu}\nabla\phi + W(u_{R,\mu})\phi\right\} \mathrm{d}x = 0, \text{ for all } \phi \in C^1(\overline{\Omega_{R,\mu}}) \text{ and } u_{R,\mu} \in W^{1,2}(\Omega_{R,\mu}).$$

The solution provided by the theorem above is classical except at most at the corners.

Second, Corollary 4.1 implies that we can assume

(5.16)
$$u_{R,\mu}(\Omega_{R,\mu}^+) \subseteq D,$$

where $\Omega_{R,\mu}^+ = \Omega_{R,\mu} \cap \{x_1 \geq 0\}$, $D = \{u_1 \geq 0\}$. Because of this, we see that a lower bound on $W(u_{R,\mu}(x))$ in $\Omega_{R,\mu}^+$ is implied by a bound from below on $\rho_{R,\mu}^+(x)$. We notice now that such a lower bound on a sufficiently large subset of $\Omega_{R,\mu}^+$ would imply

$$\int_{\Omega_{R,\mu}} W(u_{R,\mu}(x))\mathrm{d}x \geq \text{const} \cdot R^2,$$

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which in turn would run the risk of conflicting with the upper bound (3.5). We expect therefore, $\rho_{R,\mu}^+(x)$ small in $\Omega_{R,\mu}^+$ except possibly near $x_1 = 0$. From the preceding reasoning we should be able to deduce, via estimate (5.10), that $\rho_{R,\mu}^+(x)$ stays small and does not 'touch' r at $x_1 = \eta R$ for large R and μ. This is our strategy.

Proof. In what follows we write u for $u_{R,\mu}$, ρ for $\rho_{R,\mu}$ etc. Consider the sets $j_R \subset i_R \subset \mathbb{R}$ with

$$i_R := \Big\{x_1 \in (0, \eta R) \mid \text{there exists } x_2 \in (-R, R) \text{ with } \rho(x_1, x_2) \geq \frac{r}{2}\Big\}$$

and

$$j_R := \Big\{x_1 \in i_R \mid \text{there exists } x_2 \in (-R, R) \text{ with } \rho(x_1, x_2) \leq \frac{r}{4}\Big\}.$$

The positivity property (5.16) implies the lower bound

$$(5.17) \qquad 2Rw_0|i_R \setminus j_R| \leq \iint_{(i_R \setminus j_R)(-R,R)} W(u)\, \mathrm{d}x_1 \mathrm{d}x_2,$$

where

$$w_0 := \min_{\substack{|u-a| \geq r/4 \\ u \in D}} W(u) > 0.$$

From the definition of j_R we conclude that for $x_1 \in j_R$ there is an interval $L_{x_1} = (a_{x_1}, b_{x_2})$ of x_2-values such that

$$\frac{r}{4} = \rho(x_1, a_{x_1}) \leq \rho(x_1, x_2) \leq \rho(x_1, b_{x_2}) = \frac{r}{2}, \quad \text{for all } x_2 \in L_{x_1}.$$

It follows that

$$(5.18) \qquad \int_{L_{x_1}} W(u(x_1, \tau))\, \mathrm{d}\tau \geq w_0 |L_{x_1}|, \quad \text{for all } x_1 \in j_R.$$

Moreover, we have

$$(5.19) \qquad \begin{aligned} \frac{r}{4} &\leq \int_{L_{x_1}} \Big|\frac{\partial \rho}{\partial x_2}(x_1, \tau)\Big|\, \mathrm{d}\tau \leq \Big(|L_{x_1}| \int_{L_{x_1}} \Big|\frac{\partial \rho}{\partial x_2}(x_1, \tau)\Big|^2 \mathrm{d}\tau\Big)^{\frac{1}{2}} \\ &\leq \Big(|L_{x_1}| \int_{L_{x_1}} |\nabla u(x_1, \tau)|^2\, \mathrm{d}\tau\Big)^{\frac{1}{2}}. \end{aligned}$$

From (5.18) and (5.19),

$$\frac{1}{32}\frac{1}{|L_{x_1}|}r^2 + w_0|L_{x_1}| \leq \int_{L_{x_1}} \frac{1}{2}|\nabla u(x_1, \tau)|^2\, \mathrm{d}\tau + \int_{L_{x_1}} W(u(x_1\tau))\, \mathrm{d}\tau.$$

Thus,

$$(5.20) \qquad \frac{r\sqrt{w_0}}{2\sqrt{2}} \leq \int_{L_{x_1}} \frac{1}{2}|\nabla u(x_1, \tau)|^2\, \mathrm{d}\tau + \int_{L_{x_1}} W(u(x_1, \tau))\, \mathrm{d}\tau.$$

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To conclude,

$$CR \overset{(3.5)}{\geq} 2 \int_{\Omega_{R,\mu}^+} \left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\} \mathrm{d}x \geq \int_{-R}^{R} \int_{i_R} \left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\} \mathrm{d}x_1 \mathrm{d}x_2$$

(5.21)
$$= \int_{-R}^{R} \int_{i_R \setminus j_R} \left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\} \mathrm{d}x_1 \mathrm{d}x_2 + \int_{-R}^{R} \int_{j_R} \left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\} \mathrm{d}x_1 \mathrm{d}x_2$$

$$\overset{(5.17),\,(6.1)}{\geq} 2\left(2Rw_0|i_R \setminus j_R| + \frac{r_0\sqrt{w_0}}{2\sqrt{2}}|j_R|\right).$$

(5.21) implies

$$|j_R| \leq \frac{\sqrt{2}}{r\sqrt{w_0}}CR; \quad |i_R| - |j_R| \leq \frac{C}{4w_0}.$$

Therefore, it follows that

$$|i_R| \leq \frac{C}{\sqrt{w_0}}\left(\frac{1}{4\sqrt{w_0}} \frac{\sqrt{2}}{r}R\right) \leq \frac{C}{2w_0}R, \quad \text{for} \quad R \geq \frac{\mu}{4\sqrt{2w_0}}.$$

Consequently, if we take R large in (5.21), we obtain $|i_R| \leq (\sqrt{2}CR/r|w_0|) =: \eta_0 R$. If we take $\eta > \eta_0$ and fix it, then $|i_R| < \eta R$ and therefore there is an $\bar{x}_1 \in (0, \eta R)$ which does not belong to i_R and such that

(5.22)
$$\rho(\bar{x}_1, x_2) < \frac{r}{2}, \quad \text{for all } x_2 \in (-R, R).$$

Applying now the chop-off lemma for the choice $A = \{(x_1, x_2) \mid \bar{x}_1 \leq x_1 \leq \bar{\bar{x}}_1, \; |x_2| < R\}$ with $\bar{\bar{x}}_1 > \eta R$ such that $\rho(\bar{\bar{x}}_1, x_2) < r/2$ (which always exists by Lemma 5.1 for $\mu > \eta$ large enough and fixed), we conclude that $\rho \leq r/2$ in A, and thus $\rho < r$ on the line $x_1 = \eta R$.

Thus, the minimizer of the constrained problem satisfies (5.13). To conclude, we now note that $u(x) \not\equiv 0$ by the constraint. For excluding that

(5.23)
$$u(x) \not\equiv e_0(x_1), \; (e_0(0) = 0),$$

we argue as in the proof of Theorem 10 below. Specifically, estimate (7.4) holds by the same argument. Thus, the proof of the theorem is complete. □

6 The uniform exponential bound [h_1, h_2, h_3]

The hypothesis (h_3) allows the construction of a comparison function. Combining this with the exponential estimate in Corollary 5.1, we can derive, by a simple iteration argument, the sought estimate on $|u_R - a^\pm|$.

Proposition 6.1. [$\mathbf{h_1}$, $\mathbf{h_2}$, $\mathbf{h_3}$] *Suppose $r < r_0$, and η and μ (finite or infinite) fixed as in the definition of the constrained problem in Section 3. We denote the minimizer $u_{R,\mu}$ in (3.1) and the domain $\Omega_{R,\mu}$ by u_R and Ω_R respectively and assume that it possesses the property in Corollary 4.1.*
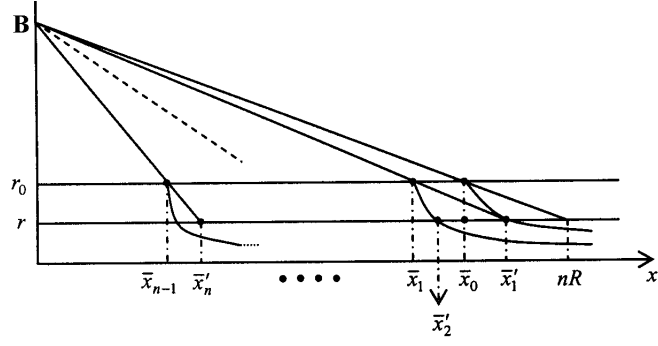
Figure 8:

Then, there exists $R_0 > 0$, such that for $x \in \Omega_R$, the estimate

$$(6.1) \qquad |u_R(x) - a^+| < M\mathrm{e}^{-c|x_1|}, \ \text{for } x_1 \geq 0, \ R \geq R_0,$$

holds, where M is a constant depending on the set C_0 in (h_2).

Corollary 6.1. *Under the hypotheses of Proposition 6.1, there exists R_0 such that*

$$(6.2) \qquad |u_R(x) - a_\pm| < r, \ \text{for } x \in \bar{C}^\pm_{R,\mu,\eta}.$$

Thus, $u_R(x)$ solves $\Delta u - W_u(u) = 0$ in Ω_R, with homogeneous Neumann conditions on $\partial\Omega_R$.

We note that Corollary 6.1 establishes the result stated in Theorem 5.1 in a different and much simpler way under the additional global hypothesis (h_3).

Proof. Step 1. We begin by noting that by Lemma 4.1 we may assume that $u_R(x) \in C_0$.

Step 2. Suppose $Q(u)$ a C^2 convex function as in (h_3). We can check easily that the following holds true

$$(6.3) \qquad \Delta_x Q(u(x)) = \mathrm{tr}\{(\partial^2 Q)(\nabla u)(\nabla u)^\top\} + Q_u(u(x)) \cdot \Delta_x u(x) \geq Q_u(u(x)) \cdot \Delta_x u(x).$$

Step 3. Let u_R be the minimizer. Then,

$$(6.4) \qquad Q(u_R(x)) \leq Ax_1 + B =: U(x_1, \eta R), \ \text{for } x_1 \in [0, \eta R], \ x = (x_1, x_2),$$

where $A = \dfrac{r - B}{\eta R}$, B a bound, and $Q(u_R(x)) \leq B$ for $x \in \Omega_R$, provided by Step 1.

To prove (6.4), from (6.3) in $\Omega_R \cap \{0 \leq x_1 \leq \eta R\}$, we have

$$\Delta_x Q\big(u_R(x)\big) \geq Q_u\big(u_R(x)\big) \cdot W_u\big(u_R(x)\big) \geq 0$$

by (6.3), (4.6), (h_3); (6.4) now follows by the maximum principle.

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We shall denote by $U(x_1; \theta)$ the function $\dfrac{r - B}{\eta} x_1 + B$. Then, $Q(u_R(x)) \leq U(x_1; \eta R)$ for $0 \leq x_1 \leq \eta R =: \bar{x}_0'$. Next, we consider the equation

(6.5)
$$U(x_1; \eta R) = r_0,$$

which has the unique solution

$$\bar{x}_0 = \frac{B - r_0}{B - r} \eta R = \delta \eta R, \text{ with } \delta = \frac{B - r_0}{B - r}, \ 0 < \delta < 1.$$

By the definition of Q, $\rho_R^+ \leq r_0$, $\bar{x}_0 \leq x_1 \leq \mu R$ from which we obtain, via Lemma 5.1, for $\bar{x}_0$ in the place of ηR,

(6.6)
$$\rho_R^+(x) \leq r_0 \frac{\cosh\left(c(\mu R - x_1)\right)}{\cosh\left(c(\mu R - \bar{x}_0)\right)} =: r_0 \sigma(x_1; \bar{x}_0), \text{ for } \bar{x}_0 \leq x_1 \leq \mu R.$$

Evaluating (6.6) at $x_1 = \eta R$, we obtain for $\eta > \delta/(1 - \delta)$,

$$\rho_R^+(x) \leq 2 r_0 e^{-c\delta R}.$$

The estimate forces, for R and η large enough, the strict inequality $\rho_R^+(x) < r$ and so by the remark following Corollary 5.1, u_R solves the Neumann problem. Consequently, Corollary 6.1 holds. Note that (6.6) is independent of μ and thus it holds for $\mu = \infty$.

Now we continue the iteration. Let $\bar{x}_1'$ be the solution to $r_0 \sigma(x_1; \bar{x}_0) = r$. As before, we have $Q(u_R(x)) \leq U(x_1; \bar{x}_1')$ for x_1 in $\in [0, \bar{x}_1']$, and therefore $\rho_R^+(x) \leq r_0$ for $x_1 \in [\bar{x}_1, \mu R]$, where $\bar{x}_1$ the solution to $U(x_1; \bar{x}_1') = r_0$. Consequently, we have the estimate

$$\rho_R^+(x) \leq r_0 \sigma(x_1; \bar{x}_1), \text{ for } \bar{x}_1 \leq x_1 \leq \mu R.$$

We denote the solution to $r_0 \sigma(x_1; \bar{x}_1) = r$ by $\bar{x}_2'$ and keep going, thus generating two sequences $\{\bar{x}_i\}, \{\bar{x}_i'\}$, for $i = 1, 2, \ldots$

The iteration is terminated if for some i, the slope of the line $U(x_1; \bar{x}_i')$ which is $(r - B)/\bar{x}_i'$, gets equal or less than $-c r_0$, the lower bound of the slope of $r_0 \sigma(x_1; \bar{x}_i)$ at the point $\bar{x}_i$. Consequently, since $\bar{x}_i'$ is decreasing as $i \to \infty$ and

$$\left| \frac{\mathrm{d}}{\mathrm{d} x_1} \Big|_{\bar{x}_i'} \sigma(x_1; \bar{x}_i) \right| \leq c,$$

we may let $i \to \infty$. The iteration is terminated independently of R and at a distance

$$\lim_{i \to \infty} \bar{x}_i' = \frac{B - r}{c r_0} =: \delta^*$$

from the line $x_1 = 0$. Moreover, we have

$$\rho_R^+(x) \leq r_0 \sigma(x_1; \lim_{i \to \infty} \bar{x}_i), \quad \lim_{i \to \infty} \bar{x}_i \leq x_1 \leq \mu R,$$

from which it follows that $\rho_R^+(x) \leq r$ for $x_1 \geq \delta^*$ and $x_1 \leq \mu R$. Thus,

$$\rho_R^+(x) \leq r_0 \frac{\cosh\left(c(\mu R - x_1)\right)}{\cosh\left(c(\mu R - \delta^*)\right)}, \text{ for } \delta^* \leq x_1 \leq \mu R.$$

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It follows that $\rho_R^+(x) \leq 2r_0 e^{-c(x_1-\delta^*)}$, for $x_1 \geq \delta^*$.

Note that

$$R_0 = -\frac{\ln\left(\dfrac{r}{2r_0}\right)}{c\delta}, \quad \delta = \frac{B-r_0}{B-r}, \quad \delta^* = \frac{B-r}{cr_0}.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

7 Taking the limit $[\mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3}, \mathbf{h_4}]$

In this section we will work with the infinite strip which we denote by Ω_R. The constrained problem in Section 3 provides a minimizer u_R which may be assumed to possess the positivity property of Corollary 4.1. Moreover, u_R satisfies the uniform exponential bound (6.1). By standard local estimates, the following limit exists

$$(7.1) \qquad\qquad u(x) = \lim_{R_n \to \infty} u_{R_n}(x),$$

along a subsequence of $R_n \to \infty$, and clearly u satisfies equation (1.1). The danger is that u may be a trivial solution. One *a priori* possibility is that $u(x) \equiv 0$ (zero-dimensional solution) which cannot be excluded on the basis of symmetry. The uniform exponential estimate however excludes this case since u satisfies the estimate

$$|u(x) - a^+| < M\mathrm{e}^{-c|x_1|}, \text{ for } x_1 \geq 0.$$

A second danger is posed by the 'scalar' one-dimensional trajectory e_0. This will be excluded by an argument utilizing (h$_4$).

Theorem 7.1. *There exists a solution u to*

$$\Delta u - W_u(u) = 0, \text{ for } u : \mathbb{R}^2 \to \mathbb{R}^2,$$

which is $\mathcal{H}_2^2$-equivariant and satisfies the estimate

$$|u(x) - a^+| < M\mathrm{e}^{-c|x_1|}, \text{ for } x_1 \geq 0.$$

Moreover,

$$(7.2) \qquad\qquad u(x) \not\equiv 0$$

and

$$(7.3) \qquad\qquad u(x) \not\equiv e_0(x_1), \ (e_0(0) = 0),$$

where $e_0(x_1)$ is the scalar connection (see (h$_4$)).

Proof. We only need to establish (7.3). We will proceed by contradiction. Thus, we assume that $u(x) = e_0(x_1)$.

Step 1. (Upper bound). There exists C independent of R such that

$$(7.4) \qquad\qquad J_{\Omega_R}(u_R) \leq C + 2E(e_\pm)R,$$

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where
$$J_{\Omega_R}(v) = \int_{\Omega_R} \left\{ \frac{1}{2}|\nabla v|^2 + W(v) \right\} \mathrm{d}x.$$

To prove (7.4), we consider the function

$$\tilde{u}(x_1, x_2) = \begin{cases} e_+(x_1), \ \text{for } x_2 \geq 1, \\ \left(\dfrac{1+x_2}{2}\right)e_+(x_1) + \left(\dfrac{1-x_2}{2}\right)e_-(x_1), \ \text{for } |x_2| \leq 1, \\ e_-(x_1), \ \text{for } x_2 \leq -1, \end{cases}$$

where $e_+(x_1) = (e_+^1(x_1), e_+^2(x_1))$, $e_-(x_1) = (e_-^1(x_1), e_-^2(x_1))$.

First we show that $\tilde{u}$ is equivariant. There holds that $T_1\tilde{u}(x) = \tilde{u}(T_1x)$, by the equivariance of $e_\pm(x_1)$, and

$$\begin{aligned} T_2\tilde{u}(x) &= T_2\left[\left(\frac{1+x_2}{2}\right)e_+(x_1) + \left(\frac{1-x_2}{2}\right)e_-(x_1)\right] \\ &= \left(\frac{1+x_2}{2}\right)e_-(x_1) + \left(\frac{1-x_2}{2}\right)e_+(x_1) \\ &= \tilde{u}(T_2x), \end{aligned}$$

where we utilized $T_2e_+(x_1) = e_-(x_1)$. Thus equivariance has been checked.

Note that $\tilde{u}$ satisfies the constraint $|\tilde{u}(x_1, x_2) - a^+| \leq r$ for $x_1 \geq \eta R$. Indeed, this follows from $e_\pm(x_1) \to a^+$, $x_1 \to +\infty$. Consequently,

(7.5)
$$J_{\Omega_R}(u_R) \leq J_{\Omega_R}(\tilde{u}).$$

Next we estimate

$$\begin{aligned} J_{\Omega_R}(\tilde{u}) &= \iint_{\Omega_R} \left\{ \frac{1}{2}|\nabla\tilde{u}|^2 + W(\tilde{u}) \right\} \mathrm{d}x_1\mathrm{d}x_2 \\ &= \left(\iint_{\Omega_R \cap \{|x_2| \leq 1\}} + \iint_{\Omega_R \cap \{|x_2| \geq 1\}} \right) \left\{ \frac{1}{2}|\nabla\tilde{u}|^2 + W(\tilde{u}) \right\} \mathrm{d}x_1\mathrm{d}x_2 \\ &\leq C' + 2(R-1)E(e_\pm) = C + 2RE(e_\pm). \end{aligned}$$

Step 2. (Lower bound).

(7.6)
$$\iint_{\Omega_R} \left\{ \frac{1}{2}\left|\frac{\partial u_R}{\partial x_1}\right|^2 + W(u_R) \right\} \mathrm{d}x_1\mathrm{d}x_2 \geq 2E(e_\pm)R.$$

To prove (7.6), we set $V_{x_2}(x_1) = u_R(x_1, x_2)$. From (6.1), $|V_{x_2}(x_1) - a^+| < Me^{-c|x_1|}$, for $x_1 \geq 0$ (with an analogous estimate for $x_1 \leq 0$). By (h_4), the minimizing property of the connection [5], and the exponential estimate, we conclude

(7.7)
$$E(V_{x_2}) \geq E(e_\pm).$$

Integrating this inequality with respect to x_2, we obtain

$$\int_{|x_2|<R} E(V_{x_2})\mathrm{d}x_2 \geq \int_{|x_2|<R} E(e_\pm)\mathrm{d}x_2$$

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which is equivalent to (7.6).

Step 3.

(7.8)
$$\iint_{\Omega_R} \left|\frac{\partial u_R}{\partial x_2}\right|^2 \mathrm{d}x_1 \mathrm{d}x_2 < C,$$

where C is a constant independent of R.

To prove (7.8), let

$$
\begin{aligned}
C + 2E(e_\pm)R &\overset{(7.4)}{\geq} J_{\Omega_R}(u_R) \\
&= \iint_{\Omega_R} \frac{1}{2}\left|\frac{\partial u_R}{\partial x_2}\right|^2 \mathrm{d}x_1 \mathrm{d}x_2 + \iint_{\Omega_R} \left\{\frac{1}{2}\left|\frac{\partial u_R}{\partial x_1}\right|^2 + W(u_R)\right\} \mathrm{d}x_1 \mathrm{d}x_2 \\
&\geq \iint_{\Omega_R} \frac{1}{2}\left|\frac{\partial u_R}{\partial x_2}\right|^2 \mathrm{d}x_1 \mathrm{d}x_2 + 2E(e_\pm)R,
\end{aligned}
$$

where in the last inequality we utilized (7.6). The idea of the proof of (7.8) is taken from [1].

Step 4. For $d > 0$ arbitrary but fixed, we have

(7.9) $\quad \displaystyle\lim_{n\to\infty} \iint_{|x_2|<d} \left\{\frac{1}{2}|\nabla u_{R_n}|^2 + W(u_{R_n})\right\} \mathrm{d}x_1 \mathrm{d}x_2 = \iint_{|x_2|<d} \left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\} \mathrm{d}x_1 \mathrm{d}x_2.$

To prove (7.9), let

$$W(u_{R_n}(x)) \to W(u(x)) \text{ pointwise}$$

and, by (6.1),

$$W(u_{R_n}(x)) < C e^{-c|x_1|}$$

Consequently, by the dominated convergence theorem

$$\lim_{n\to\infty} \iint_{|x_2|<d} W(u_{R_n}) \mathrm{d}x_1 \mathrm{d}x_2 = \iint_{|x_2|<d} W(u) \mathrm{d}x_1 \mathrm{d}x_2.$$

The corresponding statement for the gradient term follows from the exponential estimate above and the equation

$$
\begin{cases}
\Delta u_{R_n} - W_u(u_{R_n}) = 0, & \text{in the interior of } \Omega_{R_n}, \\
\dfrac{\partial u_{R_n}}{\partial n} = 0, & \text{on the boundary of } \partial\Omega_{R_n},
\end{cases}
$$

which holds by Corollary 6.1. From (7.9) and the contradiction hypothesis it follows that

(7.10) $\quad \displaystyle\lim_{n\to\infty} \iint_{|x_2|<d} \left\{\frac{1}{2}|\nabla u_{R_n}|^2 + W(u_{R_n})\right\} \mathrm{d}x_1 \mathrm{d}x_2 = 2dE(e_0).$ $\qquad$ (h$_4$)

Step 5 (Conclusion).

$$C + 2E(e_\pm)R_n \overset{(7.4)}{\geq} J_{\Omega_{R_n}}(u_{R_n}) = \iint_{|x_2|<R_n} \left\{\frac{1}{2}\left|\frac{\partial u_{R_n}}{\partial x_1}\right|^2 + \frac{1}{2}\left|\frac{\partial u_{R_n}}{\partial x_2}\right|^2 + W(u_{R_n})\right\} \mathrm{d}x_1 \mathrm{d}x_2.$$

We fix $d > 0$, arbitrary, and we write the integral above as follows.

$$\iint_{|x_2|<d} \left\{ \frac{1}{2}\left|\frac{\partial u_{R_n}}{\partial x_1}\right|^2 + \frac{1}{2}\left|\frac{\partial u_{R_n}}{\partial x_2}\right|^2 + W(u_{R_n}) \right\} \mathrm{d}x_1 \mathrm{d}x_2$$

$$+ \iint_{R_n > |x_2| > d} \frac{1}{2}\left|\frac{\partial u_{R_n}}{\partial x_1}\right|^2 \mathrm{d}x_1 \mathrm{d}x_2 + \iint_{R_n > |x_2| > d} \left\{ \frac{1}{2}\left|\frac{\partial u_{R_n}}{\partial x_1}\right|^2 + W(u_{R_n}) \right\} \mathrm{d}x_1 \mathrm{d}x_2$$

$$\geq 2dE(e_0) + \mathrm{o}(1;d) + A(d) + \iint_{R_n > |x_2| > d} \left\{ \frac{1}{2}\left|\frac{\partial u_{R_n}}{\partial x_1}\right|^2 + W(u_R) \right\} \mathrm{d}x_1 \mathrm{d}x_2,$$

where $\lim_{n\to\infty} \mathrm{o}(1;d) = 0$ for every d fixed. Also, we have the estimate $|A(d)| < C$ by (7.8), C independent of d. Thus, calling $(*)$ the left-hand side of the inequality above, we have

$$(*) \geq 2dE(e_0) + \mathrm{o}(1;d) + A(d) + 2\int_d^{R_n} \left\{ \int_{-\infty}^{+\infty} \left\{ \frac{1}{2}\left|\frac{\partial u_{R_n}}{\partial x_1}\right|^2 + W(u_{R_n}) \right\} \mathrm{d}x_1 \right\} \mathrm{d}x_2$$

$$\geq 2dE(e_0) + \mathrm{o}(1;d) + A(d) + 2(R_n - d)E(e_\pm).$$

Therefore,

$$C + 2E(e_\pm)R_n \geq 2dE(e_0) + \mathrm{o}(1;d) + A(d) + 2(R_n - d)E(e_\pm),$$

$$C + 2dE(e_\pm) \geq 2dE(e_0) + A(d) + \mathrm{o}(1;d),$$

$$\frac{C}{d} + 2E(e_\pm) \geq 2E(e_0) + \frac{A(d)}{d} + \frac{\mathrm{o}(1;d)}{d}.$$

We take now $n \to \infty$ (d fixed) to conclude that

(7.11) $$\frac{C}{d} + 2E(e_\pm) \geq 2E(e_0) + \frac{A(d)}{d}.$$

However, $A(d)$ is bounded by a constant independent of d and C is independent of d. Therefore, (7.11) leads to a contradiction by taking d large in relation to the constants above and the difference $E(e_0) - E(e_\pm)$. The proof of Theorem 7.1 is complete. $\square$

Theorem 7.1 suggests that the solution we have constructed is not trivial. Our next result is a direct statement on the two-dimensionality of this solution.

Theorem 7.2. *Let u be as in Theorem 7.1. There exists a sequence $x_2^n \to \infty$ such that*

(7.12) $$u(x_1, x_2^n) \to \tilde{e}_+(x_1), \quad u(x_1, -x_2^n) \to \tilde{e}_-(x_1),$$

where $\tilde{e}_\pm$ connecting orbits of a^+, a^- symmetric to each other, $\tilde{e}_\pm(0) = 0$ and distinct from the scalar e_0.

Recall that in contrast to [1], we do not assume uniqueness of the pair of connections minimizing the action. We generally expect multiple solutions and also that for a given solution, the limit as $x_2 \to \infty$ exists.

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Proof. There exists a C independent of R such that

(7.13) $$J_{\Omega_R}(u) \leq C + 2E(e_\pm)R$$

To prove (7.13), choose $R' > R$ but otherwise arbitrary. By the additivity of the integral,

(7.14) $$J_{\Omega_{R'}}(u_{R'}) = J_{\Omega_R}(u_{R'}) + \iint_{R \leq |x_2| \leq R'} \left\{ \frac{1}{2}|\nabla u_{R'}|^2 + W(u_{R'}) \right\} \mathrm{d}x.$$

We set $V_{x_2} := u_{R'}(x_1, x_2)$ and observe that

$$\iint_{R \leq |x_2| \leq R'} \left\{ \frac{1}{2}|\nabla u_{R'}|^2 + W(u_{R'}) \right\} \mathrm{d}x \geq \int_{R \leq |x_2| \leq R'} E(V_{x_2}) \, \mathrm{d}x_2 \geq 2E(e_\pm)(R' - R).$$

Utilizing this inequality together with (7.4) for R', we get from (7.14),

$$C + 2E(e_\pm)R' \geq J_{\Omega_R}(u_{R'}) + 2E(e_\pm)(R' - R),$$

or,

$$C' + 2E(e_\pm)R \geq J_{\Omega_R}(u_{R'}).$$

From this, (7.13) follows by taking the limit $R' \to \infty$, via the uniform exponential estimate.

The following bound follows immediately from (7.8).

(7.15) $$\iint_{\mathbb{R}^2} \left| \frac{\partial u}{\partial x_2} \right|^2 \mathrm{d}x \leq C.$$

To conclude, by elliptic theory, it follows easily from (7.15) (see [1], Lemma 5.2) that up to subsequences x_2^n, the limits $e(x_1) = \lim_{n \to \infty} u(x_1, x_2^n)$ exist and therefore solve the equation $\Delta u - W_u(u) = 0$. By the uniform exponential estimate (6.1), $e(x_1)$ is a connecting orbit of a^+ and a^-.

Consequently, in order to complete the proof, it suffices to show that $e(x_1) \neq e_0(x_1)$, which is a genuine refinement of (7.3). We proceed by contradiction. Assume that $e(x_1) \equiv e_0(x_1)$. We set

$$V_{x_2}(x_1) := u(x_1, x_2).$$

By the uniform exponential estimate (6.1) which holds for u, the contradiction hypothesis gives

(7.16) $$E(V_{x_2}) \to E(e_0), \text{ as } |x_2| \to \infty.$$

For $\varepsilon > 0$, we select R_0 such that

(7.17) $$E(V_{x_2}) \geq E(e_0) - \varepsilon, \text{ for } |x_2| \geq R_0.$$

Let $R > R_0$ arbitrary and fixed. Integrating (7.17), we get

$$\int_{|x_2| < R} E(V_{x_2}) \, \mathrm{d}x_2 \geq \int_{|x_2| < R_0} E(V_{x_2}) \, \mathrm{d}x_2 + \int_{R_0 \leq |x_2| < R} (E(e_0) - \varepsilon) \, \mathrm{d}x_2,$$

or, equivalently,

$$\iint_{|x_2| < R} \left\{ \frac{1}{2} \frac{\partial u}{\partial x_1}^2 + W(u) \right\} \mathrm{d}x_1 \mathrm{d}x_2 \geq \int_{|x_2| < R_0} E(V_{x_2}) \, \mathrm{d}x_2 + 2(E(e_0) - \varepsilon)(R - R_0),$$

which, via (7.13), is in contradiction, as $R \to \infty$, to (h$_4$). The proof of Theorem 7.2 is complete. $\qquad\square$

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8 Higher-dimensional extensions

For Extension 1 in Section 2, under the hypotheses there, we establish the following

Theorem 8.1. *There is a solution u of the system*

$$\Delta u - W_u(u) = 0, \text{ for } u : \mathbb{R}^3 \to \mathbb{R}^3, \tag{8.1}$$

where u is $\mathcal{H}_3^2$-equivariant and satisfies the estimate

$$|u(x) - a^+| < M e^{-c|x_1|}, \text{ for } x_1 \geq 0. \tag{8.2}$$

Moreover, if

$$u(x_1, x_2, \pm x_3) \to p_\pm(x_1, x_2) = \left(p_\pm^1(x_1, x_2), p_\pm^2(x_1, x_2), p_\pm^3(x_1, x_2)\right), \text{ as } x_3 \to \infty, \tag{8.3}$$

then $p_\pm$ are solutions to (8.1) satisfying (8.2), with $p_-^3 = -p_+^3 \not\equiv 0$. An analogous statement holds for $u(x_1, \pm x_2, x_3) \to q_\pm(x_1, x_3)$, as $x_2 \to \infty$.

Note that the estimates in the proof provide compactness. Thus, either the limit as $x_3 \to \infty$ exists, and so (8.3) and its analogs hold, or else there are distinct limits along distinct sequences. In either case, we have strong evidence of the three-dimensional nature of the solution.

Proof. The procedure we follow is analogous to the proof of Theorem 7.1. We work with a minimizer of the constrained problem. Here $\Omega_R = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x_2| < R, |x_3| < R\}$. As before, we can assume that u_R has the positivity property (4.16) from which the uniform exponential estimate (8.2) follows. By elliptic estimates, we have that along a sequence $R_n \to \infty$ the limit

$$u(x) = \lim_{R_n \to \infty} u_{R_n}(x) \tag{8.4}$$

exists. Symmetry excludes that $u(x) \equiv a^\pm$ and the exponential estimate excludes that $u(x) \equiv 0$.

The new element in the proof is the existence of the two-dimensional solutions that pose a threat to the existence of a genuine three-dimensional solution.

Step 1. (Two-dimensional solutions). Let

$$\tilde{W}(u_1, u_2) := W(u_1, u_2, 0)$$

and consider the problem

$$\begin{cases} \dfrac{\partial^2 \sigma_1}{\partial x_1^2} + \dfrac{\partial^2 \sigma_1}{\partial x_2^2} - \tilde{W}_u(\sigma_1) = 0, \\[2mm] \lim_{x_1 \to \pm\infty} \sigma_1(x_1, x_2) = (\pm a, 0) \end{cases} \tag{8.5}$$

in the class of $\mathcal{H}_2^2$-equivariant vector fields. For applying Theorems 7.1 and 7.2 we need to verify the hypotheses in Section 1. (h_4) is the only hypothesis that requires discussion. We

note that by (h_d), $\tilde{Q}(u_1, u_2) := Q(u_1, u_2, 0)$ satisfies the requirement since $W_{u_3}(u_1, u_2, 0) = 0$ by symmetry. Thus, the two-dimensional theory above applies and produces a solution σ_1 to (8.5), $\sigma_1 : \mathbb{R}_x^2 \to \mathbb{R}_u^2$, satisfying (by the uniqueness assumptions in (h_c)),

$$(8.6) \qquad \lim_{x_2 \to \pm\infty} \sigma_1(x_1, x_2) = e_\pm^1(x_1).$$

Similarly, we define

$$\tilde{\tilde{W}}(u_1, u_3) := W(u_1, 0, u_3)$$

and obtain a solution $\sigma_2 : \mathbb{R}_x^2 \to \mathbb{R}_u^2$ to

$$(8.7) \qquad \begin{cases} \dfrac{\partial^2 \sigma_2}{\partial x_1^2} + \dfrac{\partial^2 \sigma_2}{\partial x_3^2} - \tilde{\tilde{W}}_u(\sigma_2) = 0, \\[2mm] \lim_{x_1 \to \pm\infty} \sigma_2(x_1, x_3) = (\pm a, 0), \quad \lim_{x_3 \to \pm\infty} \sigma_2(x_1, x_3) = e_\pm^2(x_1). \end{cases}$$

By (7.13), the following estimates hold.

$$(8.8) \qquad \begin{cases} \displaystyle\iint_{|x_2| < d} \left\{ \frac{1}{2} |\nabla \sigma_1|^2 + W(\sigma_1) \right\} \mathrm{d}x_1 \mathrm{d}x_2 \leq C + 2E_I d, \\[4mm] \displaystyle\iint_{|x_3| < d} \left\{ \frac{1}{2} |\nabla \sigma_2|^2 + W(\sigma_2) \right\} \mathrm{d}x_1 \mathrm{d}x_3 \leq C + 2E_I d. \end{cases}$$

Next, we construct two two-dimensional solutions, σ_3^* and σ_4^*, with range in the $u_2 = u_3$ and the $u_2 = -u_3$ planes respectively and with corresponding asymptotic limits e^3, e^4, as $|x_2| \to \infty$.

The discussion that follows could be compressed in a couple of sentences concerning the covariance of the gradient under linear transformations and the invariance of the Laplacian under orthogonal transformations. In more detail, set

$$\{f_1, f_2, f_3\} := \big\{ (1, 0, 0), (0, 1, 0), (0, 0, 1) \big\}$$

$$\{f_1', f_2', f_3'\} = \left\{ (1, 0, 0), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}$$

and let $u = Ru'$, $R \in O(\mathbb{R}^3)$, that transforms f-coordinates to f'-coordinates. Set

$$\hat{W}(u') = W(R^\top u)$$

and note that (8.1) transforms into

$$(8.9) \qquad \Delta u' - \hat{W}_{u'}(u') = 0.$$

We can moreover change to x' independent variables, $x = Rx'$, by utilizing the invariance of the Laplacian under orthogonal transformations. Applying then Theorems 7.1 and 7.2 in the (x', u') setup, we obtain the existence of a two-dimensional $\mathcal{H}_2^2$-equivariant solution $\sigma_3^* = \sigma_3^*(x_1', x_2')$, as above, with asymptotic limits

$$(8.10) \qquad \begin{cases} \lim_{x_1' \to \pm\infty} \sigma_3^*(x_1', x_2') = (a^\pm, 0) \\[2mm] \lim_{x_2' \to \pm\infty} \sigma_3^*(x_1', x_2') = e_\pm^3(x_1') \end{cases}$$

Finally, we change back to the original independent and dependent variables

$$\hat{\sigma}_3^*(x) = \sigma_3^*(x')$$

and note that $\hat{\sigma}_3^*$ is $\mathcal{H}_2^2$-equivariant in x. Overall, we obtain the existence of two solutions which, by abusing the notation and dropping the hat, we denote by

$$(x_1, x_2) \to \sigma_3^*(x_1, x_2), \quad (x_1, x_2) \to \sigma_4^*(x_1, x_2),$$

which are $\mathcal{H}_2^2$-equivariant and map the (x_1, x_2)-plane into the $u_2 = u_3$ and $u_2 = -u_3$ planes respectively. Moreover, they satisfy the estimates

(8.11)
$$\begin{cases} \displaystyle\iint_{|x_2|<d} \left\{ \frac{1}{2}|\nabla\sigma_3^*|^2 + W(\sigma_3^*) \right\} dx_1 dx_2 \le C + 2E_{\mathrm{II}}d, \\[2ex] \displaystyle\iint_{|x_3|<d} \left\{ \frac{1}{2}|\nabla\sigma_4^*|^2 + W(\sigma_4^*) \right\} dx_1 dx_3 \le C + 2E_{\mathrm{II}}d, \end{cases}$$

with asymptotic limits

(8.12)
$$\begin{cases} \displaystyle\lim_{x_1 \to \pm\infty} \sigma_3^*(x_1, x_2) = (a^\pm, 0, 0), \quad \lim_{x_2 \to \pm\infty} \sigma_3^*(x_1, x_2) = e_\pm^3(x_1), \\[2ex] \displaystyle\lim_{x_1 \to \pm\infty} \sigma_4^*(x_1, x_2) = (a^\pm, 0, 0), \quad \lim_{x_2 \to \pm\infty} \sigma_4^*(x_1, x_2) = e_\pm^4(x_1). \end{cases}$$

Step 2. (The list of solutions). So far we have the following solutions.

- Zero-dimensional: $u(x) \equiv (a, 0, 0)$, $u(x) \equiv (-a, 0, 0)$, $u(x) \equiv (0, 0, 0)$. Among these, only $u(x) \equiv (0, 0, 0)$ is $\mathcal{H}_3^2$-equivariant but it is excluded by the exponential estimate.

- One-dimensional: Among the nine connections, e_0 is the only equivariant one and will be excluded as a possible (trivial) three-dimensional candidate by $E(e_0) > E_{\mathrm{I}}$.

- Two-dimensional: $u(x) = \sigma_1(x_1, x_2)$, $u(x) = \sigma_2(x_1, x_2)$, $u(x) = \sigma_3^*(x_1, x_2)$, $u(x) = \sigma_4^*(x_1, x_2)$. Among these, σ_1, σ_2 are the only $\mathcal{H}_3^2$-equivariant ones when extended as maps from (x_1, x_2, x_3) to (u_1, u_2, u_3). They will be excluded by the hypothesis $E_{\mathrm{I}} > E_{\mathrm{II}}$.

Step 3. (Upper bound). There is a C, independent of R, such that

(8.13)
$$J_{\Omega_R}(u_R) \le C + E_{\mathrm{II}}(2R)^2.$$

To prove (8.13), consider the function

$$\tilde{u}(x_1, x_2, x_3) = \begin{cases} \sigma_3^*(x_1, x_2), \text{ for } x_3 \ge 1, \\[2ex] \left(\dfrac{1+x_3}{2}\right)\sigma_3^*(x_1, x_2) + \left(\dfrac{1-x_3}{2}\right)\sigma_4^*(x_1, x_2), \\[2ex] \sigma_4^*(x_1, x_2), \text{ for } x_3 \le -1. \end{cases}$$

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It can be easily checked that $\tilde{u}$ is equivariant and satisfies the constraint, hence

$$J_{\Omega_R}(u_R) \le J_{\Omega_R}(\tilde{u}).$$

From this, (8.13) follows via (8.11).

Step 4. (Lower bound).

$$(8.14) \qquad \int_{\Omega_R} \left\{ \frac{1}{2}\left|\frac{\partial u_R}{\partial x_1}\right|^2 + W(u_R)\right\} \mathrm{d}x_1\mathrm{d}x_3 \ge (2R)^2 E_{\mathrm{II}}.$$

To prove (8.14), set $V_{x_2 x_3}(x_1) := u_R(x_1, x_2, x_3)$. By the exponential estimate and the hypothesis (h_c), especially (2.5), we obtain $E(V_{x_2 x_3}) \ge E_{\mathrm{II}}$. (8.14) now follows by integration.

Step 5. (Gradient estimate).

$$(8.15) \qquad \int_{\Omega_R} \left\{ \left(\frac{\partial u_R}{\partial x_2}\right)^2 + \left(\frac{\partial u_R}{\partial x_3}\right)^2\right\} \mathrm{d}x \le C.$$

The proof of (8.15) is immediate from (8.13), (8.14).

Step 6. ($u \ne \sigma_1, \sigma_2$) We proceed by contradiction, hence assume that $u(x) = \sigma_1(x_1, x_2)$. For $d > 0$ arbitrary and fixed, we have, via the exponential estimate,

(8.16)
$$\lim_{n\to\infty} \iiint_{\substack{|x_2|<d \\ |x_3|<d}} \left\{ \frac{1}{2}|\nabla u_{R_n}|^2 + W(u_{R_n})\right\} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = 2d \iint_{|x_2|<d} \left\{ \frac{1}{2}|\nabla \sigma_1|^2 + W(\sigma_1)\right\} \mathrm{d}x_1\mathrm{d}x_2.$$

Now,

$$C + E_{\mathrm{II}}(2R_n)^2 \ge J_{\Omega_{R_n}}(u_{R_n}),$$

where $J_{\Omega_{R_n}}(u_{R_n})$ is expanded as

$$\iiint_{\substack{|x_2|<d \\ |x_3|<d}} \left\{ \frac{1}{2}|\nabla u_{R_n}|^2 + W(u_{R_n})\right\} \mathrm{d}x + \iiint_{\substack{d<|x_2|<R_n \\ d<|x_3|<R_n}} \left\{ \frac{1}{2}\left(\frac{\partial u_{R_n}}{\partial x_2}\right)^2 + \frac{1}{2}\left(\frac{\partial u_{R_n}}{\partial x_3}\right)^2\right\} \mathrm{d}x$$

$$+ \iiint_{\substack{d<|x_2|<R_n \\ d<|x_3|<R_n}} \left\{ \frac{1}{2}\left(\frac{\partial u_{R_n}}{\partial x_2}\right)^2 + W(u_{R_n})\right\} \mathrm{d}x + \{\text{ rest }\}$$

and thus bounded by

$$
\begin{aligned}
(8.17) \qquad &\ge (2d)^2 E_{\mathrm{I}} \qquad \text{(by the contradiction hypothesis and } (h_c)) \\
&+ A(d) \qquad (A(d) < C \text{ by } (7.16)) \\
&+ 4\big((R_n - d)^2 + 2(R_n - d)d\big)E_{\mathrm{II}} \qquad \text{(by lower-bound estimates)} \\
&+ o(1; d).
\end{aligned}
$$

From (8.17) we obtain a contradiction as in (7.11).

So far we have established the analogue of Theorem 7.1 for (8.1). Next we proceed to establish the analog of Theorem 7.2.

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Step 7. There exists C independent of R, such that

$$(8.18) \qquad J_{\Omega_R}(u) \leq C + E_{\mathrm{II}}(2R)^2.$$

This follows from (8.13). See the derivation of (8.2).

Step 8. For a constant C independent of R, we have

$$(8.19) \qquad \iiint_{\mathbb{R}^3} \left\{ \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2 \right\} \mathrm{d}x \leq C.$$

which follows immediately from (8.15).

Step 9. (Conclusion). Assume that

$$u(x_1, x_2, \pm x_3) \to p_\pm(x_1, x_2), \text{ as } x_3 \to \infty.$$

By equivariance, $p_-^3 = -p_+^3$. Suppose, for the sake of contradiction, that $p_\pm^3 = 0$. By the exponential estimate, we deduce that $E(u(\cdot, x_2, x_3)) \to E(p_+(\cdot, x_2))$, as $x_3 \to \infty$, uniformly in x_2. Since $p_+(x_1, x_2) \in \{u_3 = 0\}$,

$$E(p_+(\cdot, x_2)) \geq E_\mathrm{I}, \text{ by } (\mathrm{h_c}).$$

Hence, given $\varepsilon > 0$, there is an R_0, such that for $R > R_0$ and arbitrary otherwise, we have

$$\int_{|x_3|<R} E(u(\cdot, x_2, x_3)) \, \mathrm{d}x_3 \geq \int_{|x_3|<R_0} E(u(\cdot, x_2, x_3)) \, \mathrm{d}x_3 + 2(E_\mathrm{I} - \varepsilon)(R - R_0),$$

and therefore, by integrating in x_2, we obtain

$$\iint_{\substack{|x_2|<R \\ |x_3|<R}} E(u(\cdot, x_2, x_3)) \, \mathrm{d}x_2\mathrm{d}x_3 \geq \iint_{\substack{|x_2|<R \\ |x_3|<R_0}} E(u(\cdot, x_2, x_3)) \, \mathrm{d}x_2\mathrm{d}x_3 + (2R)(E_\mathrm{I} - \varepsilon)(R - R_0).$$

This last estimate contradicts (2.5) for large R. Hence, we reached a contradiction, so p_+^3 cannot be identically zero.

The proof of Theorem 8.1 is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

For Extension 2 in Section 2, under the hypotheses following there, we establish the following

Theorem 8.2. *There is a bounded G-equivariant solution u of the system*

$$(8.20) \qquad \Delta u - W_u(u) = 0,$$

satisfying

$$(8.21) \qquad \lim_{x_1 \to \pm\infty} u(x_1, x') = (\pm\alpha, 0'),$$

which is nontrivial in the sense that there exists a connection e minimizing the action E, such that, for any unit vector $\nu \in F' \cap \{x_1 = 0\}$, we have

$$(8.22) \qquad \lim_{\lambda \to +\infty} u(x_1, \lambda r\nu) = re(x_1), \text{ for all } x_1 \in \mathbb{R}, r \in R'.$$

(Without loss of generality we have assumed that $e(\mathbb{R}) \subset F'$.)

Sketch of proof. The argument is analogous to the proof of Theorem 7.1. In particular, one can derive the following upper and lower bounds for the energy $J(u, Q_L)$ of a minimizer of the constrained problem. We denote by Q_L the cube

$$Q_L := \{x \in \mathbb{R}^n \mid -L < x_j < L, \ j = 1, \ldots, n\}$$

and by u_L a minimizer of the constrained problem. Then,

$$(8.23) \qquad J(u_L, Q_L) \leq (2L)^{n-1}(1 - e^{-\frac{k_0}{L}})m + k_1\lambda^{n-2},$$

$$(8.24) \qquad J(u, Q_L) \geq (2L)^{n-1}(1 - e^{-\frac{k_0}{L}})m,$$

where m is the minimum of the action among the connecting trajectories. $\qquad \square$

Condition (8.22) implies in particular that the entire solution u in Theorem 8.3 cannot be represented with a function $v : \mathbb{R}^m \to \mathbb{R}^n$ with $m < n$. More precisely, for any given $v : \mathbb{R}^{n-1} \to \mathbb{R}^n$ and $2 \leq j \leq n$, there is an x such that

$$(8.25) \qquad u(x_1, \ldots, x_n) \neq v(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

As a corollary to the theorem above we obtain the following

Theorem 8.3. *Assume $W : \mathbb{R}^2 \to \mathbb{R}$ satisfies all the assumptions in Theorem 8.2 for $n = 2$ and define $V : \mathbb{R}^3 \to \mathbb{R}^3$ by setting*

$$(8.26) \qquad V(u_1, u_2, u_3) := W(u_1, \sqrt{u_2^2 + u_3^2}).$$

Then, there exists an entire solution u of the system

$$(8.27) \qquad \Delta u - V_u(u) = 0$$

which satisfies the asymptotic condition

$$(8.28) \qquad \lim_{\lambda \to +\infty} u(x_1, \lambda\rho_\theta\nu) = \rho_\theta e(x_1), \ \text{for all } x_1 \in \mathbb{R}, \ \theta \in [0, 2\pi),$$

for some $e \in E$, where $\rho_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation by θ around the x_1-axis.

Sketch of proof. Given $L > 1$, let Q_L be the cube

$$Q_L := \{x \in \mathbb{R}^3 \mid -L < x_j < L, \ j = 1, 2, 3\}.$$

Let u_L be a minimizer of $J(\cdot, Q_L)$. Define the comparison function $\hat{u}_L : Q_L \to \mathbb{R}^3$ by setting

$$(8.29) \qquad \hat{u}_L(x_1, \lambda\rho_\theta\nu) = \rho_\theta e(x_1), \ \text{for } \lambda > 1$$

$$(8.30) \qquad \hat{u}_L(x_1, \lambda\rho_\theta\nu) = \lambda\rho_\theta e(x_1), \ \text{for } 0 \leq \lambda \leq 1.$$

The definition of $\hat{u}_L$ implies

$$(8.31) \qquad J(\hat{u}_L, Q_L) \leq 4L^2(1 - e^{-\frac{k_0}{L}})m + k_1 \ln \lambda,$$

for some constants $k_0, k_1 > 0$. Since e is a minimizer of the connection problem, we also have

$$(8.32) \qquad J(u, Q_L) \geq 4L^2(1 - e^{-\frac{k_0}{L}})m$$

for all equivariant maps $u : Q_L \to \mathbb{R}^3$ that satisfy the constraints. $\qquad \square$

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